DEPOSITORY
10 NOV 1977

NAVAL RESEARCH LOGISTICS QUARTERLY

SEPTEMBER 1977
VOL. 24, NO. 3

OFFICE OF NAVAL RESEARCH

NAVSO P-1278

NAVAL RESEARCH LOGISTICS QUARTERLY

EDITORS

Murray A. Geisler
Logistics Management Institute

W. H. Marlow
The George Washington University

Bruce J. McDonald
Office of Naval Research

MANAGING EDITOR

Seymour M. Selig
Office of Naval Research
Arlington, Virginia 22217

ASSOCIATE EDITORS

Marvin Denicoff
Office of Naval Research

Alan J. Hoffman
IBM Corporation

Neal D. Glassman
Office of Naval Research

Jack Laderman
Bronx, New York

Thomas L. Saaty
University of Pennsylvania

Henry Solomon
The George Washington University

The Naval Research Logistics Quarterly is devoted to the dissemination of scientific information in logistics and will publish research and expository papers, including those in certain areas of mathematics, statistics, and economics relevant to the over-all effort to improve the efficiency and effectiveness of logistics operations.

Information for Contributors is indicated on inside back cover.

The Naval Research Logistics Quarterly is published by the Office of Naval Research in the months of March, June, September, and December and can be purchased from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. Subscription Price: $11.15 a year in the U.S. and Canada, $13.95 elsewhere. Cost of individual issues may be obtained from the Superintendent of Documents.

The views and opinions expressed in this Journal are those of the authors and not necessarily those of the Office of Naval Research.

Issuance of this periodical approved in accordance with Department of the Navy Publications and Printing Regulations, P-35 (Revised 1-74).

OPTIMAL DISPOSAL POLICIES FOR A SINGLE-ITEM INVENTORY SYSTEM WITH RETURNS

Daniel P. Heyman

Bell Telephone Laboratories
Holmdel, New Jersey

ABSTRACT

We consider a single-item inventory system in which the stock level can increase due to items being returned as well as decrease when demands occur. Returned items can be repaired and then used to satisfy future demand, or they can be disposed of. We identify those inventory levels where disposal is the best policy.

It is shown that this problem is equivalent to a problem of controlling a single-server queue. When the return and demand processes are both Poisson, we find the optimal policy exactly. When the demand and return processes are more general, we use diffusion approximations to obtain an approximate model, which is then solved. The approximate model requires only mean and variance data. Besides the optimal policy, the output of the models includes such characteristics as the operating costs, the purchase rate for new items, the disposal rate for returned items and the average inventory level. Several numerical examples are given.

An interesting by-product of our investigation is an approximation for the steady-state behavior of the bulk GI/G/1 queue with a queue limit.

0. INTRODUCTION

Operating telephone companies provide their customers with station equipment (e.g., a telephone set or a data set) as part of satisfying the customer's telecommunications needs. When the customer terminates a particular service, the appropriate piece of apparatus is returned to the company. A returned item may be repaired so that it is "as good as new," placed in inventory, and then used to satisfy future demand. Thus, for many types of equipment, telephone company inventories exhibit upward fluctuations due to returned items as well as the usual downward fluctuations caused by demands.

For such an inventory system, a natural question to ask is "when is the inventory level so high that retaining another returned item is uneconomical because the cost of repairing the item and holding it in inventory until it can be used is greater than the savings that can be obtained by satisfying a future demand with a repaired item rather than a new item?" For high-cost, low-volume items this question is particularly significant because inventory carrying charges are typically considered to be proportional to both the purchase price of the item and the length of

time it is held in inventory, and the low demand rate can result in an item being held in inventory for a long time.

Fukuda [4] considers disposal policies in a multiechelon inventory system, but his model does not contain returns. Tainiter [13] and Whisler [14] consider models for rented equipment that account for returned items; they describe profit functions and find optimal consumer behavior respectively. As in [13] and [14] we shall make use of some correspondences that can be drawn between inventory and queuing processes. Schrady [12] presents a deterministic model of an inventory system with returns of repairable items and determines the optimal procurement and repair quantities.

In Section 1 we formulate the problem of setting the maximum stock level. In Section 2 we show that this formulation is equivalent to a problem in the control of queues. This allows us to bring the accumulated knowledge of queueing theory to bear on our problem. In Section 3 we solve the model when the returns and demands form Poisson processes. In Section 4 we drop the Poisson assumptions and obtain approximate solutions to the general problem using recently developed diffusion model approximations for queues, [7] and [8.] Numerical examples are given and discussed in Section 5.

1. MODEL FORMULATION

We will consider a single-product inventory system. At epochs $\xi_1, \xi_2, \ldots$ requests for the product are made; we assume that time intervals $X_1=\xi_1, X_i=\xi_{i+1}-\xi_i, i=2, 3, \ldots$ are nonnegative independent and identically distributed (i.i.d) random variables. At epoch ξ_i the random amount U_i is demanded, where $U_1, U_2, \ldots$ are nonnegative independent random variables and U_i is independent of X_j for all i and j. Let $\{D(t), t\geq 0\}$ denote the total number of items demanded at time t. With the above assumptions, $\{D(t), t\geq 0\}$ is a *cumulative process* with independent nonnegative increments [3]. We can identify two interesting special cases of this type of process; if the U_i are identically equal to one, $D(t)$ is a renewal process, while if the X_i are identically equal to a constant, $D(t)$ represents a process where random demands arrive at equally spaced intervals. We take X and U as the generic forms for X_i and U_i respectively.

These demands can be equally well satisfied by repairing old items or by purchasing new ones. We assume that repairable items are returned according to the cumulative process $\{R(t), t\geq 0\}$, where Y denotes the generic time interval between return epochs, and W denotes the generic number returned at a return epoch. Thus, $R(t)$ denotes the cumulative number of items returned.

We assume that repair and purchase lead times are negligible. The consequences of removing this assumption are discussed in Section 6.

The assumption that repair lead times are negligible is particularly crucial. A partial justification for this assumption is that it may be possible to expedite repairs occasionally. Then, whenever the stock of repaired items is close to zero, some (or all) of the unrepaired items are rushed through the repair process so that with high probability no demands are lost. This is tantamount to assuming that repairs can be expedited so that there are no demand epochs with some unrepaired but no repaired items on hand.

If there is no on-shelf inventory at a demand epoch, the demand is immediately satisfied by a purchase. Since purchase lead times are zero, there are no back orders or lost sales. Purchases of new items are made as required and not in lots because, at least for typical telephone equipment, set-up costs for ordering are relatively small.

Let r be the repair cost per item, p be the purchase cost per item, h be the cost rate for holding an item in inventory, and β be the discount rate (continuous discounting). A returned item that is not kept is called a *disposed* item. Disposed items may be sold to another company, converted to another piece of equipment, sold for salvage, etc. We let d denote the net receipts from disposing of an item; a negative value of d will represent a net loss of disposal. To avoid trivialities we assume

$$p-r-d>0,$$

so that it is less expensive to repair an item than to dispose of one and purchase another.

At each return epoch we have the option of disposing of some (or all) of the returned items. Once an item is placed in inventory it cannot be disposed of. Our decision variable, denoted by N, is the maximum number of items allowed in inventory. Returns that would raise the inventory level above N are disposed of; all other returns are kept. The number N is called the *keep level*. In Section 2.3 we will give conditions for such a single critical number policy to be optimal.

Let $C_N(t)$, $p_N(t)$, $r_N(t)$ and $d_N(t)$ denote the total cost, the total number of purchases, repairs, and disposals, respectively, at time t using the keep level N, and $I_N(t)$ be the inventory level. Then,

$$C_N(t)=p\cdot p_N(t)+r\cdot r_N(t)-d\cdot d_N(t)+h\int_0^t I_N(u)du \tag{1-1}$$

The asymptotic expected cost rate $C(N)$ is defined by

$$C(N)=\lim_{T\to\infty} C_N(T)/T \tag{1-2}$$

and the asymptotic expected discounted cost $C(N;\beta)$ is defined by

$$C(N;\beta)=\lim_{T\to\infty}\int_0^T e^{-\beta t}dE[C_N(t)]. \tag{1-3}$$

The purpose of this paper is to find those values of N, N^* and N_β^* say, that minimize (1–2) and (1–3) respectively. For those cases where we cannot obtain exact answers, approximate answers will be sought.

In order to make our problem readily amenable to mathematical analysis, we assume that $D(t)$ and $R(t)$ are independent. This assumption should be valid whenever the number of actual users of the product is small compared to the number of potential users. Then, the decision of a user to make a return is likely to be independent of a potential user's decision to request an item.

2. FORMULATION AS A QUEUEING DESIGN PROBLEM

In this section we will formulate the problem of setting a keep level as the problem of selecting the queueing capacity (maximum queue size) of a single-server queue. This allows us to relate the vast literature and available methodology for single-server queueing problems to our problem.

2.1 The Equivalent Queueing Problem

Let $I_N(t)$ be the inventory level at time t; a typical realization of $I_N(t)$ is shown in Figure 1 below. If $I_N(t)<N$, it increases at return epochs, and if $I_N(t)>0$, it decreases at demand epochs. Now define $S_N(t)$ by

$$S_N(t)=N-I_N(t), \qquad t\geq 0. \tag{2-1}$$

Corresponding to the realization of $I_N(t)$ shown in Figure 1, a realization of $S_N(t)$ is given in Figure 2.

Figure 2 looks exactly like the sample path of a single-server queue with a maximum system (the number in queue plus in service) size of N. The arrival epochs when the queue is not full behave as demand epochs that are satisfied by repaired stock since downward jumps in $I_N(t)$ correspond to upward jumps in $S_N(t)$. Furthermore, when $S_N(t)=N$, e.g., the interval (t_3, t_4), arrivals are rejected. This is precisely an interval where $I_N(t)=0$ and demands are satisfied by purchases. Thus, we make the further correspondence between rejected arrivals in the queueing model and demands that are satisfied by purchases in the inventory model. Similarly, the service-completion epochs behave as return epochs where the returned item is not disposed because downward jumps in $S_N(t)$ correspond to upward jumps in $I_N(t)$. Let us assume that during the idle period of the queueing system (i.e., when $S_N(t)=0$) the server operates on an infinite supply of secondary jobs

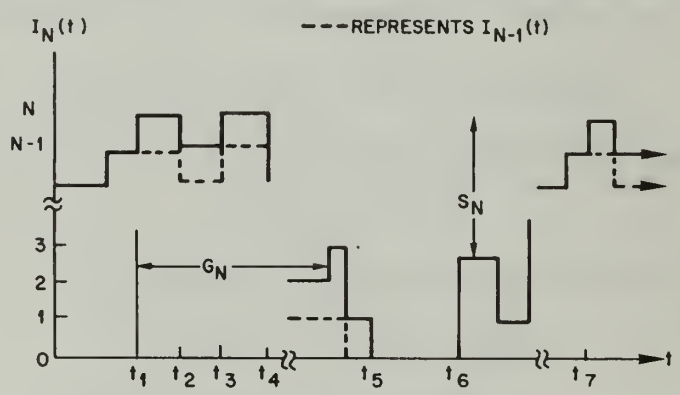


FIGURE 1. A typical realization of the inventory level.

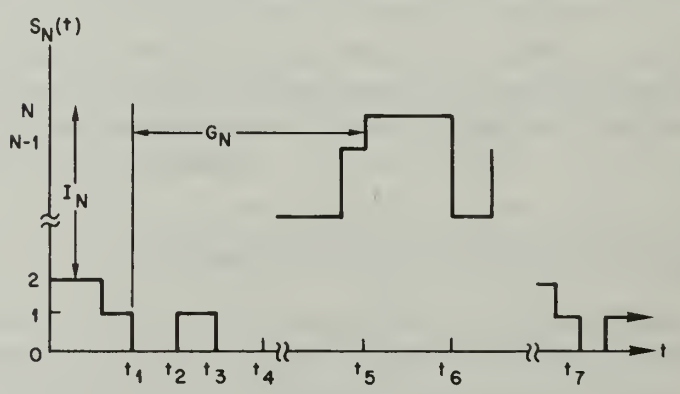


FIGURE 2. The corresponding realization of $S_N(t)$.

and processes them in the same fashion it processes primary jobs, processing the secondary jobs terminates when a primary job arrives, and is started anew at the beginning of the next idle period. In the language of priority queueing, this queue discipline is called *preemptive repeat.* Then, since idle periods in the queueing model correspond to periods when returns are disposed of, the number of secondary jobs processed in the queueing model corresponds to the number of disposed returns in the inventory model. Note that in the queueing model, $I_N(t)$ corresponds to the number of idle positions in the system (see Figure 1).

If the queueing system is changed r for every primary job completed, p for every primary customer rejected because all positions are occupied, the cost rate h is charged on every idle place in the system, and earns d for every secondary job completed, then the problem of choosing the optimal size of the queueing system is identical to choosing the optimal keep level for the inventory system.

Having identified the correspondence between the inventory and queueing problems, we will use the standard queueing symbols to describe both problems. Since we will always have a single-server system with N–1 places in the queue, we will suppress this information in our notation, e.g., the system where both demand and return epochs constitute Poisson processes will be an M/M system. We shall also use the terminology of the inventory and queueing models interchangeably.

2.2 Consequents from the Queueing Model

The queueing model just described has a general bulk-arrival process, a general bulk-service mechanism, a single-server and a finite system capacity; the standard symbolic notation for this type of queue is $G_U/G_W/1/N$. For certain special cases (see Cohen [1]) some analytical results have been obtained, but the only bounded queueing structure that yields results that are easily manipulated is the $M/M/1/N$ queue. In this case, both $S_N(t)$ and $I_N(t)$ are Markov processes, and this fact allows us to analyze them readily. For this model we will obtain the exact value of N^* and some exact results for the operating characteristics of the system. Our purpose in obtaining these results is to develop certain properties of this special structure that may be expected to be approximately true for more general cases, and to provide a benchmark for testing the approximations for the general model that will be developed later.

Further results that can readily be obtained from the queueing model are that for the general inventory process, time intervals when the inventory is at its' maximum level correspond to idle periods in the queueing system, e.g., $[t_1, t_2)$ and $[t_3, t_4)$ in Figures 1 and 2, hence they are i.i.d. Time intervals when the inventory is less than its maximum level correspond to busy periods in the queueing system, e.g., $[t_2, t_3)$ and $[t_4, t_7)$ in Figures 1 and 2, hence they are i.i.d. Since an idle period is independent of the preceding busy period when the times between arrival epochs have a negative-exponential distribution, when the inventory process becomes full, its future evolution is independent of its past when the times between demand epochs have a negative-exponential distribution. Finally, the time intervals determined by successive departures from the all inventory positions full condition correspond to consecutive busy cycles in the queueing system, e.g., $[t_2, t_4)$ and $[t_4, t_8)$ in Figures 1 and 2, hence they are i.i.d. and, moreover, their end points form a set of regeneration points for $I_N(t)$.

2.3 Optimality of the Single Critical Number Policy

When the demand process is compound Poisson, the inventory process when looked at only at return epochs (which are our decision points) is a semi-Markov process. Hence our decision problem can be formulated as a Markov-renewal program [9]. For ease of presentation only, let us consider only undiscounted problems where the returns arrive one by one.

Let the state of the decision process be the number of items in inventory at a return epoch. Clearly there is a finite state N' which is so large that if a returned item were accepted, the expected holding cost incurred until that item was used would be greater than the savings one could gain from satisfying a demand by a repaired item. Hence, the optimal decision in states $\geq N'$ is to dispose of returned items. Thus, we essentially have a decision process with finitely many states where two different decisions can be made in each state. For these problems it is known that a stationary policy is optimal [9].

Now we will show that using a stationary policy is equivalent to using a single critical number policy. Suppose we accept returns in states $<n$, reject them in state n, and leave the decision in states $n+1, \ldots, N'$ arbitrary, $0 \leq n \leq N'$. As long as the inventory level at time zero is finite, the inventory will (almost surely) reach state n or smaller in a finite amount of time, and from then on no state larger than n is ever observed, so the given policy is equivalent to using the single critical number policy that disposes of returns when the inventory level is n.

Thus we have shown that one can always do as well with a single critical number policy as with any other policy. When the demands are not compound Poisson, this statement need not be true because the optimal decision may depend on information about the time to the next demand epoch. However, the single critical number is as good as any policy that depends only on the inventory level at a return epoch.

3. ANALYSIS OF THE M/M MODEL

Let λ and μ be the rates of the demand (arrival) and return (service) processes respectively. Define

$$\pi_N(n;\, t)=P_r\{I_N(t)=n\},\ \pi_N(n)=\lim_{t\to\infty}\pi_N(n;\, t),\ \sigma_n(n;\, t)=P_r\{S_N(t)=n\} \text{ and } \sigma_N(n)=\lim_{t\to\infty}\sigma_N(n;\, t);$$

from (2–1) we obtain $\pi_N(n;\, t)=\sigma_N(N-n;\, t)$ for all t.

3.1 The Optimal Policy: Cost-per-Time Criterion

Since the rate at which items are purchased equals the rate at which demands arrive multiplied by the fraction of time the inventory level is zero, we obtain

$$\lim_{t\to\infty} p_N(t)/t=\lambda\pi_N(0)=\lambda\sigma_N(N) \tag{3–1}$$

By similar reasoning we also obtain

$$\lim_{t\to\infty} d_N(t)/t=\mu\pi_N(N)=\mu\sigma_N(0) \tag{3–2}$$

and

$$\lim_{t\to\infty} r_N(t)/t=\mu[1-\pi_N(N)]=\mu[1-\sigma_N(0)]. \tag{3-3}$$

Asymptotically, the fraction of time the inventory level equals $n=\pi_N(n)$, hence

$$\lim_{t\to\infty} t^{-1}\int_0^t I_N(u)du=\sum_{n=1}^{N} n\pi_N(n)=N-\sum_{n-1}^{N} n\sigma_N(n). \tag{3-4}$$

The same limiting results are also obtained if $p_N(t)$, $d_N(t)$, $r_N(t)$, and $I_N(t)$ are replaced by their expected values. Using (3-1)-(3-4) together with (1-1) and (1-2) yields

$$\begin{aligned} C(N)=&p\lambda\sigma_N(N)+r\mu[1-\sigma_N(0)]-d\mu\sigma_N(0)\\ &+h\left[N-\sum_{n=1}^{N} n\sigma_N(n)\right]. \end{aligned} \tag{3-5}$$

Define $\rho=\lambda/\mu$; ρ is called the traffic intensity and represents the average number of arrivals during a service interval. From well known results in queueing theory (see, e.g. [5], p. 66),

$$\sigma_N(n)=\begin{cases}(1-\rho)\,\rho^n/(1-\rho^{N+1}), & \rho\neq 1\\ 1/(N+1), & \rho=1\end{cases} \tag{3-6}$$

for $n=0, 1, \ldots N$, and defining $L_N=\sum_{n=1}^{N} n\sigma_N(n)$,

$$L_N=\begin{cases}\dfrac{\rho}{1-\rho^{N+1}}\,\dfrac{1-\rho^N[1+N(1-\rho)]}{1-\rho}, & \rho\neq 1\\ \dfrac{N}{2}, & \rho=1\end{cases} \tag{3-7}$$

for $N>0$.

Using (3-6) one can show directly that

$$\rho\sigma_N(N)+1-\sigma_N(0)=\rho. \tag{3-8}$$

In telephone traffic, this equation represents the conservation of load. It states that the offered load (ρ) equals the carried load ($1-\sigma_N(0)$) plus the lost load ($\rho\sigma_N(N)$) and is true under more general conditions than are given above. To interpret (3-8) for an inventory system, multuply both sides by μ to obtain

$$\lambda\sigma_N(N)+[1-\sigma_N(0)]\mu=\lambda \tag{3-9}$$

which is equivalent to

$$\lambda\pi_N(0)+[1-\pi_N(N)]\mu=\lambda. \tag{3-10}$$

The right side of (3-10) is the demand rate and the left side is the rate at which purchases are required plus the rate at which returns are added to the inventory. In the steady-state, the rate at which the inventory increases equals the rate at which it decreases, hence we may conclude from (3-10) that the demand rate equals the purchase rate plus the rate of satisfying demand from inventory, a rather obvious proposition.

Substituting (3–10) into (3–5) yields, for $N>0$,

$$C(N)=\lambda(p-r-d)\sigma_N(N)+h(N-L_N)+\lambda r-d(\mu-\lambda); \tag{3–11}$$

since the last two terms of (3–11) are independent of N, $C(N)$ is minimized whenever

$$\bar{C}(N)=\lambda(p-r-d)\sigma_N(N)+h(N-L_N), \qquad N>0, \tag{3–12}$$

is minimized, so it is sufficient to only consider $\bar{C}(N)$. When $N=0$, there will be no idle positions in the system at any time and all arrivals will be rejected, so we define

$$\bar{C}(0)=\lambda(p-r-d). \tag{3–13}$$

Substituting (3–6) and (3–7) into (3–12) yields

(3–14a)

$$\bar{C}(N)=\lambda(\rho-r-d)(1-\rho)\rho^N/(1-\rho^{N+1})+h\left[N-\frac{\rho}{1-\rho^{N+1}}\,\frac{1-\rho^N[1+N(1-\rho)]}{1-\rho}\right], \qquad \rho\neq 1$$

and

$$\bar{C}(N)=\lambda(p-r-d)/(N+1)+hN/2, \qquad \rho=1. \tag{3–14b}$$

From (3–13), (3–14a) and (3–14b) one can show, via a considerable amount of algebraic manipulation, that $\bar{C}(N)\leq\bar{C}(N-1)$, for $N\geq 1$, if and only if

$$\frac{h}{\lambda}\,\frac{\rho}{(1-\rho)^2\rho^N}\{1-\rho^N[1+N(1-\rho)]\}\leq p-r-d, \qquad \rho\neq 1 \tag{3–15a}$$

$$\frac{h}{\lambda}\,\frac{N(N+1)}{2}\leq p-r-d, \qquad \rho=1. \tag{3–15b}$$

In particular, for any positive value of ρ, $\bar{C}(1)\leq\bar{C}(0)$ if and only if

$$h/\lambda\leq p-r-d. \tag{3–16}$$

In a straightforward manner we can show that the left-hand sides of (3–15) and (3–16) are strictly increasing functions of N, hence N^* is the largest N for which (3–15) holds. From (3–16) we learn that $N^*=0$ if and only if $h/\lambda>p-r-d$. This result has the following interpretation. $1/\lambda$ is the mean time between demand epochs, so if $N=1$, h/λ is the expected holding cost of a returned item that is not disposed of. If an item is satisfied by a purchase, the cost of satisfying that item is p. By repairing a returned item, the cost of satisfying one unit of demand is r, and since the disposal value of d was not obtained, the net savings from the repaired item is $p-r-d$. Thus, (3–16) states that $N^*=0$ whenever the expected cost of holding the item in inventory is greater than the savings that can be achieved from holding the item until the next time it can be used to satisfy some demand. In the next section, (3–15a) and (3–15b) will be shown to have the same interpretation.

We summarize the important results of this section by stating them as a theorem.

THEOREM 1: For the M/M system, N^* is the largest value of N for which (3–15) holds, and $N^*=0$ if and only if (3–16) is false.

For $\rho=1$, we can find N^* in closed form. Assume (3–16) is true so $N^*>0$. Treating (3–15b) as an equation and N as a continuous variable, (3–15b) is a quadratic equation in N whose only positive solution is

$$[-1+\sqrt{1+8\lambda(p-r-d)/h}]/2 \triangleq \hat{N}$$

hence, N^* is the largest integer less than or equal to $\hat{N}$.

3.2 An Alternate Analysis of the M/M System

In this section, we derive the results of Section 3.1 in a completely different way; this will serve to elucidate the meaning of equation (3–15).

Let us define a *cost cycle* of $I_N(t)$ as a time interval that starts when $I_N(t)=N$ and ends the next time $I_N(t)=N$ and includes a point when $I_N(t)=0$, i.e., it is the first passage time of $I_N(t)$ from N to N given a visit to $I_N(\cdot)=0$. In Figure 1, $[t_1, t_7)$ is the only cost cycle. A formal definition of a cost cycle is an interval $[t', t'')$ such that: i) $I_N(t')=I_N(t'')=N$; ii) $I_N(t'-)=I_N(t''-)<N$; and iii) $I_N(t)=0$ for some $t \in (t', t'')$.

Referring to Figure 1, we see that during the cost cycle $[t_1, t_7)$ the differences between using a keep level of N and a keep level of $N-1$ are: i) at t_1 a disposal is made with keep level $N-1$ that is not made with keep level N; ii) during the interval $[t_1, t_5)$ the C-stock level is smaller by one item when the keep level is $N-1$; iii) at t_5 a purchase is made with keep level $N-1$ that is not made with keep level N; iv) during the interval (t_5, t_7), $I_N(t)=I_{N-1}(t)$.

Let G_N denote the (random) length of time from the start of a cost cycle until the stocklevel reaches zero for the first time; in Figure 1, $G_n=t_5-t_1$. The lengths of the cost cycles form a set of i.i.d. random variables; thus the renewal-reward theorems assert that the cost per unit time and the expected cost per unit time asymptotically equal the expected cost of a cost cycle divided by the expected length of a cost cycle, whenever both these quantities are finite (see, e.g., [11], p. 52). Suppose that at the start of a cost cycle we arbitrarily dispose of one item and then when the inventory reaches zero another item is purchased. Then this cost cycle would be identical to a cost cycle of a system operating with a keep level that is one unit smaller. Thus, the expected cost of a cost cycle with keep level N is no larger than the expected cost of a cost cycle with keep level $N-1$ if and only if

$$hE(G_N) \leq p-r-d, \qquad N=1, 2, \ldots, \tag{3–17}$$

and (3–17) is necessary and sufficient for $C(N) \leq C(N-1)$.

Now we obtain $E(G_N)$. Let an *event* denote a demand or a return. Since the demand and return processes are Poisson, at any point in time the probability that the next event is a demand, denoted by q, is given by $q=\lambda/(\lambda+\mu)$, and the time until the next event occurs has a negative-exponential distribution with mean $(\lambda+\mu)^{-1}$; let $\bar{q}=1-q$ denote the probability that the next event is a return. Let K_N be the number of events during G_N; K_N can be considered to be the number of events in the first passage from N to zero of a discrete random-walk process with a reflecting barrier at N. Writing $q=\rho/(1+\rho)$ and $\bar{q}=(1+\rho)^{-1}$, and using a result of Hardin and Sweet [6], we find, for $N \geq 1$,

$$E(K_N)=\begin{cases}(1+\rho)(1-\rho)^{-2}\{\rho^{-N}-[1+N(1-\rho)]\}, & \rho \neq 1 \\ N(N+1), & \rho=1.\end{cases}$$

Since the expected time between events is $(\lambda+\mu)^{-1}=\rho/[\lambda(1+\rho)]$, we obtain

$$E(G_N)=\begin{cases}\dfrac{1}{\lambda}\dfrac{\rho}{(1-\rho)^2\rho^N}\{1-\rho^N[1+N(1-\rho)]\}, & \rho\neq 1\\ \dfrac{N(N+1)}{2\lambda}, & \rho=1\end{cases} \tag{3-18}$$

for $N\geq 0$. Substituting (3–18) into (3–17) yields (3–15), which shows that (3–15) has the same interpretation as (3–16), and, of course, the two methods of analyzing this system lead to the same optimal policy. However, in the method used in this section we never obtain an explicit expression for $C(N)$ [or $\bar{C}(N)$], although this can be done with a substantial amount of analysis and algebra.

3.3 The Optimal Policy: Discounted Cost Criterion

In this section we will obtain N_β^*, the optimal keep level when costs are continuously discounted at rate β, for the M/M model. The analysis will be similar to the analysis used in Section 3.2.

Let $G_N(t)=Pr\{G_N\leq t\}$,

$$\tilde{G}_N(\beta)=\int_0^\infty e^{-\beta t}dG_N(t)$$

For any nonnegative random variable Q with $Q(t)=Pr\{Q\leq t\}$ and

$$\tilde{Q}(\beta)=\int_0^\infty e^{-\beta t}dQ(t),$$

the following properties of Laplace-Stieltjes transforms are well known:

PROPERTY 1: If a cost B is incurred Q time units from now, then the expected present worth of this cost is $B\tilde{Q}(\beta)$.

PROPERTY 2: If a cost rate of b is incurred until Q time units from now, then the expected present value of the total cost incurred is $(b/\beta)[1-\tilde{Q}(\beta)]$.

Using the analysis leading to (3–17) and the above two properties, we find that $C_N(\beta)\leq C_{N-1}(\beta)$ if and only if

$$d+[1-\tilde{G}_N(\beta)]h/\beta\leq(p-r)\tilde{G}_N(\beta)$$

or

$$\tilde{G}_N(\beta)\geq(h+d\beta)/[h+(p-r)\beta]. \tag{3-19}$$

Now we obtain a closed-form expression for $\tilde{G}_N(\beta)$. Let $\hat{K}_N(\theta)=\sum_{k=0}^{\infty}\theta^k Pr\{K_N=k\}$; Hardin and Sweet [6] showed that

$$\hat{K}_N(\theta)=(q\theta)^N(\lambda_1-\lambda_2)/F_0(\theta) \tag{3-20}$$

where

$$\lambda_1=\{1+[1-4q\bar{q}]^{1/2}\}/2,$$

$$\lambda_2=\{1-[1-4q\bar{q}]^{1/2}\}/2,$$

and

$$F_0(\theta)=(1-\bar{q}\theta)(\lambda_1^N-\lambda_2^N)-q\bar{q}\theta^2(\lambda_1^{N-1}-\lambda_2^{N-1}).$$

Since $\tilde{G}_N(\beta|K_N=k)=\left(\frac{\lambda+\mu}{\lambda+\mu+\beta}\right)^k$, we obtain

$$\tilde{G}_N(\beta)=\sum_{k=0}^{\infty}\left(\frac{\lambda+\mu}{\lambda+\mu+\beta}\right)^k Pr\{K_N=k\}$$

$$=\hat{K}_N\left(\frac{\lambda+\mu}{\lambda+\mu+\beta}\right), \tag{3-21}$$

hence (3–20) and (3–21) determine $\tilde{G}_N(\beta)$. Since G_N is the first passage time from N to zero in a random walk process, it clearly increases with N, and hence, $G_N(\beta)$ is an increasing function of N. Thus, we have proved Theorem 2 below.

THEOREM 2: For the M/M model, if $N=1$ makes (3–19) false, $N^*(\beta)=0$; otherwise, $N^*(\beta)$ is the largest N satisfying (3–19).

Note that we were able to specify the optimal policy without explicitly obtaining the cost function to be minimized.

3.4 A Limiting Result when the Interest Rate Vanishes

In this section we will show that N^* is a good approximation of N_β^* for small values of β. Let us consider (3–17) and (3–19) as equations, and replace the discrete variable N by the continuous variable x. Thus, (3–17) and (3–19) become

$$F(x)\triangleq E(G_x)-(p-r-d)/h=0 \tag{3-22}$$

and

$$F(x;\beta)\triangleq \tilde{G}_x(\beta)-(h+d\beta)/[h+(p-r)\beta]=0 \tag{3-23}$$

respectively. Since the left-hand sides of (3–22) and (3–23) are monotone functions of x, the solutions of the equations are unique; let x^* be the solution of (3–22) and $x^*(\beta)$ be the solution of (3–23). Then, $N^*=[x^*]$ and $N_\beta^*=[x_\beta^*]$, where $[\cdot]$ is the greatest integer function. We will now show that $\lim_{\beta\downarrow 0} x^*(\beta)=x^*$.

Using a Taylor's series expansion we can show that

$$\tilde{G}_x(\beta)=1-E(G_x)\beta-o(\beta); \tag{3-24}$$

substituting (3–24) into (3–23) yields

$$E(G_x)-o(\beta)/\beta-\beta(p-r-d/[h+(p-r)\beta]=0. \tag{3-25}$$

As $\beta\downarrow 0$, (3–25) approaches (3–22), which can be restated as

$$\lim_{\beta\downarrow 0}\beta^{-1}F(x;\beta)=F(x). \tag{3-26}$$

Since $F(x, \beta)$ and $F(x)$ are continuous and monotone functions of their arguments, it follows from (3–26), using straightforward methods, that $\lim_{\beta \downarrow 0} x^*(\beta) = x^*$.

This limiting result implies that if x^* is not an integer, $\lim_{\beta \downarrow 0} N^*_\beta = N^*$; moreover, there is a positive number β_0 say, such that $\beta \leq \beta_0$ implies $N^*_\beta = N^*$. If x^* is an integer, $\lim_{\beta \downarrow 0} N^* = N^* + 1$ if $\beta^{-1}F(x, \beta) \to F(x)$ from above and $\lim_{\beta \downarrow 0} N^*_\beta = N^* - 1$ if $\beta^{-1}F(x, \beta) \to F(x)$ from below.

3.5 The Discounted Cost Function

In this section we will indicate how to obtain expressions for $C(N, \beta)$ and obtain an approximation. These results are useful, e.g., in evaluating the consequences of using a nonoptimal policy.

Define the Lapace transforms

$$\tilde{\sigma}_N(n; \beta) = \int_0^\infty e^{-\beta t} \sigma_N(n; t) dt$$

and

$$\tilde{\pi}_N(n; \beta) = \int_0^\infty e^{-\beta t} \pi_N(n; t) dt.$$

From (2–1) we obtain $\tilde{\pi}_N(n; \beta) = \tilde{\sigma}_N(N-n; \beta)$.

From (1–3) and the discussion in Section 3.1, we see that $C(N; \beta)$ is determined by $\tilde{\sigma}_N(n; \beta)$, $n = 0, ., \ldots, N$. Morse [10] obtained $\sigma_N(n; t)$ from which $\tilde{\sigma}_N(n; \beta)$ can be determined, in principle, by integration; the required result may be obtained more easily by straightforward Laplace-transform analysis of the system of differential equations (see, e.g., [10, p. 64]) that determine $\sigma_N(n; t)$. The resulting expressions are sufficiently complicated that we choose not to record them here.

We can obtain a simple expression for $C(N; \beta)$ by assuming that at time zero the process $\{I_N(t), t \geq 0\}$ is in the steady state. This corresponds to assuming that the system has been operating for a long time before we measure it. With this assumption, $\pi_N(n; t) = \pi_N(n)$ for all t and

$$C(N; \beta) = C(N)/\beta.$$

Hence, an approximation for $C(N; \beta)$, $C'(N; \beta)$ say, is

(3–27a) $$C'(N; \beta) = \beta^{-1}\{[(p-r)\lambda(1-\rho)\rho^N - d\mu(1-\rho)]/(1-\rho^{N+1}) + h[N - \rho\{1-\rho^N[1+N(1-\rho)]\}/(1-\rho)(1-\rho^{N+1})]\} + r\lambda/\beta, \qquad \rho \neq 1$$

and

(3–27b) $$C'(N; \beta) = \beta^{-1}\{(p-r-d)\mu/(N+1) + hN/2\} + r\mu/\beta, \qquad \rho = 1.$$

3.6 Conclusions

The inventory problem of finding the optimal keep level was formulated as the problem of selecting the optimal number of waiting positions in a queueing system. This formulation enabled us to bring the vast knowledge of the single-server queue to bear on the problem. Realizing that conveniently used probabilistic results are available only when both the demand and return proc-

esses are Poisson, we solved that special case under the criteria of minimum asymptotic cost rate and minimum expected present value of costs.

The inequalities (3–17) and (3–19) characterizing the optimal keep levels only required the demand process to be a compound-Poisson process for their validity; the additional assumptions of demands occurring one at a time and the return process being Poisson were used only to enable us to obtain first passage time results in a simple way. Similarly, when the cost functions were derived, the special assumptions of the M/M process were only needed to obtain simple expressions for the required probabilities.

4. APPROXIMATIONS FOR THE GENERAL PROBLEM

Now we turn our attention to the general problem where the demand and return processes are arbitrary cumulative processes with independent increments. We will refer to this as the G/G problem. This is the form of the problem of most practical importance, since the special properties of the Poisson process are not likely to be valid assumptions in many situations. Since, for the $G_U/G_W/1/N$ queue, the exact probabilistic results needed for finding the optimal policies are not available, we will use approximations. These approximations will be based on the diffusion models developed in [7] and [8].

4.1 The Idea of the Approximation

To obtain our approximation, we replace the discrete-valued process $S_N(t)$ (equivalently $I_N(t)$) by a suitably chosen diffusion process, $\{s_N(t),\ t\geq 0\}$ say. Then we approximate G_N by the first passage time from 0 to N of $s_N(t)$, denoted by g_N. Formally, g_N is generically defined by

$$g_N=\inf_t \{t: s_N(\tau^-)>0,\ s_N(\tau+t)=N\}.$$

We can now use available results for diffusion processes to obtain the mean and Laplace transform (the latter denoted by $\tilde{g}_N(\beta)$) of g_N. We then substitute $E(g_N)$ and $\tilde{g}_N(\beta)$ for $E(G_N)$ and $\tilde{G}_N(\beta)$ in (3–17) and (3–19) respectively.

One of the main reasons that this method of approximation enables us to obtain analytical results is that since $\{s_N(t),\ t\geq 0\}$ is a Markov process, g_N does not depend on the previous history of $S_N(\cdot)$. But, referring to Figure 2, in general G_N depends on $S_N(t)$ for t between the last time $S_N(\cdot)=0$ and t_1.

In order to successfully apply this approximation, it must be *plausible* to approximate $\{S_N(t),\ t\geq 0\}$ by a diffusion process. Intuitively, this means that the fluctuations of $S_N(t)$ are predominantly stochastic. An example of a situation where the diffusion approximation will not yield satisfactory results is when there is one return each morning and five items are demanded on every fifth afternoon.

4.2 The Optimal Policies

From [2], p. 94, we obtain, for large values of t,

$$E[D(t)]\approx E(U)t/E(X),$$

$$E[R(t)] \approx E(W)t/E(Y),$$
$$V[D(t)] \approx [V(U)/E(X)+V(X)E^2(U)/E^3(X)]t,$$
$$V[R(t)] \approx [V(W)/E(Y)+V(Y)E^2(W)/E^3(Y)]t,$$

hence,

$$-c \triangleq \lim_{t\to\infty} E[D(t)-R(t)]/t = \frac{E(U)}{E(X)} - \frac{E(W)}{E(Y)} \tag{4-1}$$

and

$$V \triangleq \lim_{t\to\infty} V[D(t)-R(t)]/t = \frac{V(U)}{E(X)} + \frac{V(X)E^2(U)}{E^3(X)} + \frac{V(W)}{E(Y)} + \frac{V(Y)E^2(W)}{E^3(Y)}. \tag{4-2}$$

Now let the diffusion process $\{(s_N t),\ t\geq 0\}$ have infinitesimal mean and variance of c and v respectively and reflecting barriers at zero and N.

The mean and Laplace transform of g_N were obtained in [3] (which has a very confusing typographical error); the results are

$$E(g_N) = \begin{cases} vc^{-2}(e^{2cN/v}-1-2cN/v)/2, & c\neq 0 \quad (4\text{-}3a) \\ N^2/v, & c=0 \quad (4\text{-}3b) \end{cases}$$

and

$$\tilde{g}_N(\beta) = \begin{cases} \gamma e^{-cN/v}[\gamma \cosh(\delta N) - \sin h(\delta N)]^{-1}, & c\neq 0 \quad (4\text{-}4a) \\ \sqrt{2\beta/v}\,[\cosh(N\sqrt{2\beta/v}]^{-1}, & c=0 \quad (4\text{-}4b) \end{cases}$$

where $\gamma=\sqrt{1+2\beta v c^{-2}}$ and $\delta=c\gamma v^{-1}$.

Substituting (4-3) into (3-17) and (4-4) into (3-19) allows us to obtain approximations for N^* and N_β^* respectively. These results are summarized below.

For the general keep level problem, N^* is approximately given by the largest value of N satisfying

$$vc^{-2}(e^{2cN/v}-1-2cN/v)/2 \leq (p-r-d)/h, \quad c\neq 0 \tag{4-5a}$$

$$N^2/v \leq (p-r-d)/h, \quad c=0 \tag{4-5b}$$

and N_β^* is approximately given by the largest value of N satisfying

$$\gamma e^{-cN/v}[\gamma \cosh(\delta N) - \sinh(\delta N)]^{-1} \geq (h+d\beta)/[h+(p-r)\beta], \quad c\neq 0 \tag{4-6a}$$

$$\sqrt{2\beta/v}[\cosh(N\sqrt{2\beta/v})]^{-1} \geq (h+d\beta)/[h+(p-r)\beta], \quad c=0 \tag{4-6b}$$

Using the methods of Section 3.4 on (4-5) and (4-6) one can show that the same limiting (as $\beta \downarrow 0$) relationships between N_β^* and N^* hold for the approximate optimal policies for the general problem as hold for the exact optimal policies for the special case of the M/M problem. Note also that the approximate optimal policies were obtained without obtaining approximations for the cost functions.

4.3 Cost Functions and Operating Characteristics

In this section we will obtain approximations for the asymptotic expected cost rate and the expected discounted cost function. These may be useful, e.g., for examining the effects of using different keep levels.

Equation (3–5) gives $C(N)$ for the M/M case, and it is not valid for the general problem because the argument that the rate at which items are purchased equals the rate at which demands arrive multipled by the fraction of time the inventory level is zero holds only when the demands occur according to a Poisson process. Thus, (3–1) is not valid in general, and neither are (3–2) and (3–3), hence (3–5) is not valid in general.

However, it seems plausible that these results should be approximately correct for the general problem. Therefore, we will assume that (3–5) is valid for the general case too. It follows from this assumption that $C(N)$ is approximately given by

$$C(N) \doteq p\lambda\sigma_N(N)-(d+r)\mu\sigma_N(0)+h(N-L_N)+r\mu \tag{4–7}$$

where

$$\lambda=E(U)/E(X),\ \mu=E(W)/E(Y),\ \sigma_N(n)=Pr\{s_N=n\}$$

and

$$L_N=\sum_{n=0}^{N} n\sigma_N(n).$$

We now seek an approximation for $\sigma_N(n)$. For the diffusion process $\{s_N(t)\geq 0\}$ let

$$F_N(t;\ x)=Pr\{s_N(t)\leq x\}$$

and $F_N(x)=\lim_{t\to\infty} F_N(t;\ x)$. Then $F_N(t,\ x)$ satisfies the diffusion equation

$$\frac{\partial}{\partial t}F_N(t;\ x)=-c\frac{\partial}{\partial x}F_N(t;\ x)+\frac{v}{2}\frac{\partial^2}{\partial x^2}F_N(t;\ x) \tag{4–8}$$

with boundary conditions

$$F_N(t;\ 0)=0 \text{ and } F_N(t;\ N)=1, \qquad t>0. \tag{4–9}$$

If $s_N(0)=x_0$, then we must have

$$F_N(0,\ x)=\begin{cases}0, & 0\leq x<x_0\\ 1, & x_0\leq x\leq N\end{cases} \tag{4–10}$$

In the steady state, $\frac{\partial}{\partial t}F_N(t;\ x)=0$ and hence

$$F_N(x)=\begin{cases}(1-e^{2cx/v})\ (1-e^{2cN/v})^{-1}, & c\neq 0\\ x/N & c=0\end{cases} \tag{4–11}$$

where c and v are given by (4–1) and (4–2) respectively.

Now we make the approximation

$$\sigma_N(n) \doteq F_{N+1}(n+1)-F_{N+1}(n) \tag{4–12}$$

Substituting (4–11) into (4–12) yields

$$\sigma_N(n) \doteq \begin{cases}\alpha^n(1-\alpha)(1-\alpha^{N+1})^{-1}, & \alpha\neq 1\\ 1/(N+1) & \alpha=1\end{cases} \tag{4–13}$$

where $\alpha=\exp(2c/v)$. Comparing (4–13) and (3–6) we see that the approximate values of $\sigma_N(n)$ for the general problem that are given by (4–13) have the same form as the exact values of $\sigma_N(n)$ given by (3–6) for the M/M model. That is the reason the subscripts $N+1$ were used on the right side of (4–12). Notice also that when $c=0$, (4–13) is exact for the M/M model.

From (4–13) we obtain

$$L_N \doteq \begin{cases} \dfrac{\alpha}{1-\alpha^{N+1}} \dfrac{1+(N\alpha-N-1)\alpha^N}{1-\alpha}, & \alpha \neq 1 \\ N/2 \qquad\qquad\qquad\qquad , & \alpha=1. \end{cases} \tag{4–14}$$

Substituting (4–13) and (4–14) into (4–7) yields

(4–15a)

$$C(N) \doteq [p\lambda\alpha^N-(d+r)\mu](1-\alpha)(1-\alpha^{N+1})^{-1}+h\{N-\alpha[1+(N\alpha-N-1)\alpha^N](1-\alpha)(1-\alpha^{N+1})\}+r\mu$$

for $\alpha \neq 1$, and

$$C(N) \doteq [p\lambda-(d+r)\mu-h](N+1)^{-1}+hN+r\mu, \qquad \alpha=0. \tag{4–15b}$$

Substituting (4–13) into (3–1) and (3–2) yields for $\alpha \neq 1$,

$$\lim_{t\to\infty} p_N(t)/t \doteq E(x)\alpha^N(1-\alpha)/[E(U)(1-\alpha^{N+1})] \tag{4–16}$$

and

$$\lim_{t\to\infty} d_N(t)/t \doteq [E(Y)/E(W)][1-(1-\alpha)(1-\alpha^{N+1})^{-1}]; \tag{4–17}$$

which give approximations for the asymptotic rates of purchases and disposals respectively. Corresponding expressions when $\alpha=1$ are obtained similarly.

An approximation for expected discounted cost can be obtained from (4–15) in the same manner that (3–33) was obtained from (3–11), i.e., substitute (4–15) into $C'(N;\beta)=C(N)/\beta$.

By considering the case $\alpha=1$, we see from (4–5b) and (4–15b) that the approximate value of N^* does not, in general, minimize the approximate cost function. This is because of the approximations used. Based on [7] and [8], it is the author's opinion that $E(g_N)$ yields a more accurate approximation of $E(G_N)$ than (4–12) does of $\sigma_N(n)$, so that (4–5) yields a better approximation of N^* than minimizing (4–15) would.

Note that (4–13) provides an approximation to the state distribution of the $G_L/G_W/1/N$ queue, and using the methods in [7], an approximation for the mean, variance and Laplace transform of the busy period can be obtained.

5. NUMERICAL COMPARISONS BETWEEN THE EXACT AND APPROXIMATE SOLUTIONS TO THE M/M MODEL

In Section 3 we solved the M/M model exactly and in Section 4 we obtained approximate solutions to the G/G model. In order to find out how good the approximate solutions are, we will use the M/M assumptions in the G/G model and compare the approximate solutions to the exact solutions, using representative data.

We will use the following data: $p=\$1000.$, $r=\$200.$, $h=0.25$ per dollar per year, $d=\$10$, λ(mean demand)$=10$ items per week, and $\beta=\ln 1.09$, (i.e., we use an annual interest rate of 9 percent). In Tables I and II we vary the mean number of returns per week from 6 to 20. We use the symbol E for the exact model, A for the approximate model, and EA for the exact functions evaluated at the approximate keep level. The latter provides information about the sensitivity of the cost function, disposal rate, average inventory level, etc. to the discrepancy between the exact and approximate keep levels, and also indicates how well the appproximations for the operating characteristics perform.

TABLE I. *Comparison of exact and approximate models: cost rate criterion*

	Keep Level	Cost Rate ($/wk)	Average Yearly Purchases	Average Yearly Disposals	Average Inventory Level	Mean time from disposal to next purchase (weeks)
$\mu=6$/wk. E	658	5, 207.	208. 0	0.	1. 50	164. 12
A	659	5, 217.	208. 0	$.19\times10^{-69}$	3. 52	164. 25
EA	659	5, 207.	208. 0	0.	1. 50	164. 38
$\mu=8$/wk. E	332	3, 619.	104. 0	$.13\times10^{-32}$	4. 00	164. 00
A	333	3, 641.	104. 0	$.37\times10^{-14}$	8. 51	164. 25
EA	333	3, 619.	104. 0	$.45\times10^{-30}$	4. 00	164. 50
$\mu=10$/wk. E	56	2, 273.	9. 12	9. 12	28. 00	159. 60
A	57	2, 273.	8. 97	8. 97	28. 50	162. 45
EA	57	2, 273.	8. 97	8. 97	28. 50	165. 30
$\mu=12$/wk. E	22	2, 088.	1. 59	105. 59	17. 35	151. 62
A	22	2, 157.	6. 98	110. 98	14. 74	136. 39
EA	22	2, 088.	1. 59	105. 59	17. 35	151. 62
$\mu=14$/wk. E	15	2, 035.	0. 96	208. 96	12. 57	131. 50
A	16	2, 105.	5. 89	213. 89	11. 55	150. 60
EA	16	2, 036.	0. 68	208. 68	13. 56	185. 70
$\mu=20$/wk. E	9	1, 946.	0. 51	520. 51	8. 01	101. 30
A	10	2, 019.	5. 40	525. 40	7. 76	116. 72
EA	10	1, 947.	0. 25	520. 25	9. 01	203. 60

TABLE II. *Comparison of exact and approximate models: discounted cost criterion*

		Keep Level	Discounted Cost ($1000)	Average Yearly Purchases	Average Yearly Disposals	Average Inventory Level	Mean time from disposal to next purchase (weeks)
$\mu=6$/wk.	E	125	60. 43	208. 0	$0.23+10^{-25}$	1. 50	30. 88
	A	281	60. 54	208. 0	$0.21+10^{-28}$	3. 52	69. 75
	EA	281	60. 43	208. 0	$0.57+10^{-60}$	1. 50	69. 88
$\mu=8$/wk.	E	72	42. 00	104. 0	$0.88+10^{-5}$	4. 00	34. 00
	A	125	42. 25	104. 0	$0.41+10^{-4}$	8. 51	60. 25
	EA	125	42. 00	104. 0	$0.64+10^{-10}$	4. 00	60. 50
$\mu=10$/wk.	E	34	26. 78	14. 86	14. 86	17. 00	59. 50
	A	35	26. 73	14. 44	14. 44	17. 50	61. 25
	EA	35	26. 73	14. 44	14. 44	17. 50	63. 00
$\mu=12$/wk.	E	20	24. 25	2. 31	106. 31	15. 47	102. 01
	A	20	25. 23	8. 61	112. 61	13. 15	91. 62
	EA	20	24. 25	2. 31	106. 31	15. 47	102. 01
$\mu=14$/wk.	E	14	23. 63	1. 35	209. 35	11. 60	92. 86
	A	15	24. 58	7. 04	215. 04	10. 68	106. 81
	EA	15	23. 61	0. 96	208. 96	12. 57	131. 50
$\mu=20$/wk.	E	9	22. 58	0. 51	520. 51	8. 01	101. 30
	A	10	23. 43	5. 40	525. 40	7. 76	116. 72
	EA	10	22. 60	0. 25	520. 25	9. 01	203. 60

Recall that the discounted cost function was not obtained exactly, but was approximated by dividing the steady-state cost per unit time by the discount factor. Thus, the cost results for the undiscounted model given in Table I are more accurate than cost results for the discounted model given in Table II, and explains why the "exact" discounted cost for the approximate "optimal" policy may be lower than the cost of the optimal policy in Table II.

In Table I we see that the approximate values for the keep level never differ by more than one from the exact values and that the costs of optimal and nearly optimal policies are nearly equal. These properties are present in Table II when $\mu \geq 10$. In Table II when $\mu < 10$, the approximate value of N_β^* (the optimal keep level with discounting) is not close to the exact value, but the resulting average yearly disposals are extremely small for both the exact and the EA cases. Thus the exact and approximate keep levels are both essentially infinite, i.e., the inventory essentially

ever reaches the keep level. This effect can also be seen by looking at the average inventory level. When $\mu=6$ or 8, the exact and EA results for the average inventory level agree even though the exact and approximate keep levels are very different.

Observe in Table II that for $\mu<10$, the approximate value of the keep level is not close to the exact value but the approximate cost of a nonoptimal policy equals (to at least four significant places) the approximate cost of an optimal policy, and for $\mu=14$, the approximate value of the keep level is close to the exact value, but the approximate cost of a nonoptimal policy differs from the approximate cost of the optimal policy. This suggests that for $\mu<10$ the approximate discounted cost function is flat in the neighborhood of the optimal keep level and for $\mu<10$ we may expect to see some curvature. This is shown to be true in Figures 3 and 4. Note that for $\mu=6$, a keep level of 9 yields an approximate cost that is indistinguishable from that of the optimal keep level of 281. Since the approximate discounted cost functions shown in Figure 3 are a constant multiple of the exact cost-rate fuuction, the shape of the latter is also depicted in Figure 3.

In both tables we see that for $\mu\leq 8$, the optimal policy essentially states that no returns should be disposed of. Several additional calculations were made to see how large the return rate must be before a significant number of items would be disposed of per year. The average number disposed of per year reaches 0.5 when $\mu=9.6$ returnes per week for the cost/time criterion and when $\mu=9.1$ for the discounted cost criterion. Thus, some disposal may be optimal when the return rate is slightly smaller than the demand rate. This is due to the stochastic fluctuations of the returns and demands.

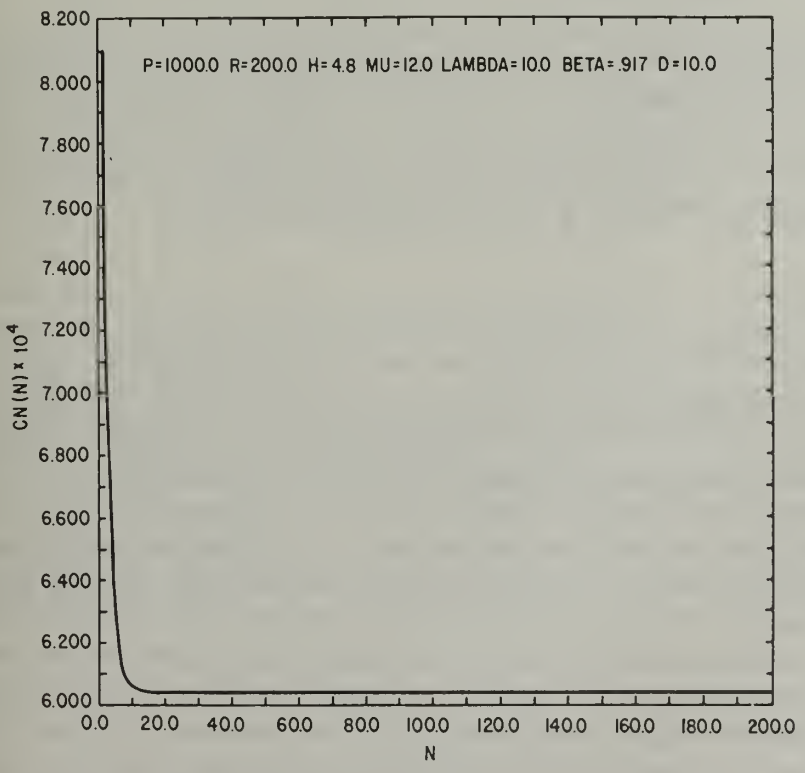


FIGURE 3. Approximate discounted cost vs. keep level, $\mu=6.0$.

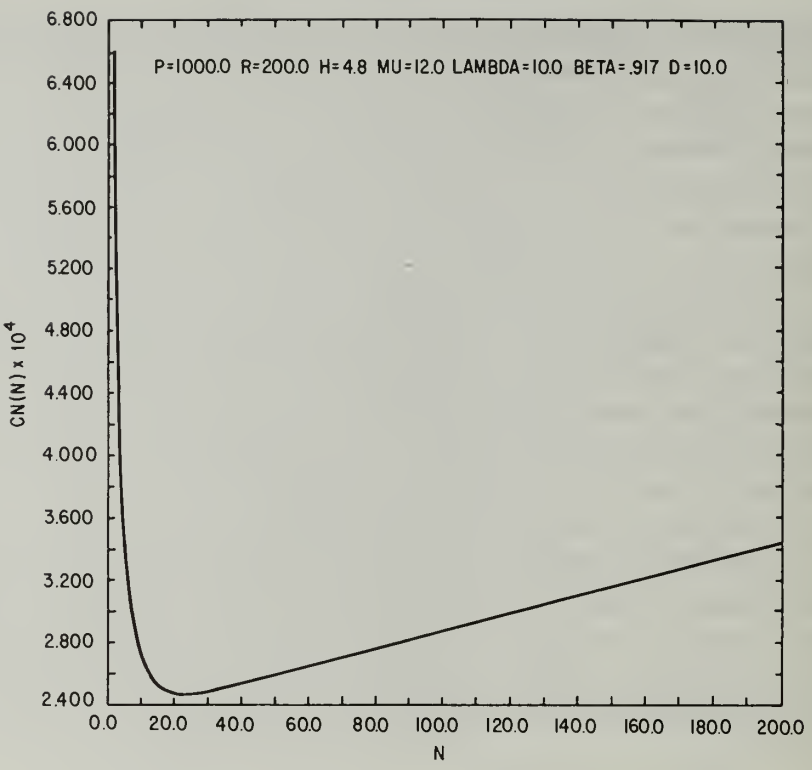


FIGURE 4. Approximate discounted cost vs. keep level, $\mu=12.0$.

Another phenomenon that appears in both tables is that the approximate policy is always as least as large as the exact policy. This was also observed in several dozen other examples, but we do not have a theoretical explanation for it, nor do we claim that it holds in general. From Figures 3 and 4 we see that it is better to use a keep level that is too high rather than one that is too low so the approximations seem to err in the direction of least harm.

6. CONCLUSIONS

We have formulated the problem of selecting maximum stock levels and shown its equivalenc to a problem in the control of queues. When the return and demand processes are Poisson, w solved the model exactly. For more general processes we used a diffusion approximation to th model and obtained approximate solutions. For both cases we considered total discounted cos and cost per time criteria. Several numerical examples were discussed; they indicated that th diffusion approximation provided good solutions when the parameters for Poisson processes wer used in it. This gives us cause to believe that the approximate model will yield adequate solution for the general problem. One important feature of the diffusion model is that it uses only the firs two moments of the return and repair processes, information that is likely to be available in practice

Two assumptions of the models that may not hold in practice are that purchase and repai lead times are negligible. Some justification for this was provided in section 1. Since deman cannot be satisfied by unrepaired items, introducing repair lead times will require describin the inventory level by two components—the number of repaired and the number of unrepaire

tems. This causes us to loose the structure of a simple single server queue, and with even the implest stochastic assumptions, the author has not been able to obtain useful exact results. However, by using a heuristically developed diffusion model similar to the one described in this paper, the problem can be analyzed approximately. A complete discussion of these issues is deferred o a subsequent paper.

ACKNOWLEDGEMENT

I would like to thank R. C. Chang of Bell Laboratories for many helpful suggestions and discussions. In particular, he developed the notion of the cost cycle and contributed several key observations leading to the analysis in Section 3.2.

REFERENCES

[1] Cohen, J. W., *The Single Server Queue* (Wiley, New York, 1969).

[2] Cox, D. R., *Renewal Theory* (Wiley, New York, 1962).

[3] Cox, D. R., and H. D. Miller, *The Theory of Stochastic Processes* (Methuen, London, England, 1965).

[4] Fukuda, Yoichiro, "Optimal Disposal Policies," Naval Research Logistics Quarterly, *8*, 221–227 (1961).

[5] Gross, Donald, and Carl M. Harris, *Fundamentals of Queuing Theory* (Wiley, New York, 1974).

[6] Hardin, J. C., and A. L. Sweet, "Moments of the Time to Absorption in the Random Walk Between a Reflecting and Absorbing Barrier," SIAM Review, *12*, 140–142 (1970).

[7] Heyman, Daniel P., "An Approximation for the Busy Period of the M/G/1 Queue Using a Diffusion Model," Journal of Applied Probability, *11*, 159–169 (1974).

[8] Heyman, Daniel P., "A Diffusion Model Approximation for the GI/G/1 Queue in Heavy Traffic," Bell System Technical Journal, *54*, 1637–1646 (1975).

[9] Jewell, William S., "Markov Renewal Programming I and II," Operations Research, *11*, 938–971 (1963).

[10] Morse, Phillip M., *Queues, Inventories and Maintenance* (Wiley, New York, 1958).

[11] Ross, Sheldon M., *Applied Probability Models with Optimization Applications* (Holden-Day, San Francisco, California, 1970).

[12] Schrady, David A., "A Deterministic Inventory Model for Repairable Items," Naval Research Logistics Quarterly, *14*, 391–398 (1967).

[13] Tainiter, M., "Some Stochastic Inventory Models for Rental Situations," Management Science, *11*, 316–326 (1964).

[14] Whisler, William D., "A Stochastic Inventory Model for Rental Equipment," Management Science, *13*, 640–647 (1967).

COMPOUND DISTRIBUTIONS WITH EFFICIENT COMPUTATION IN INVENTORY MODEL APPLICATIONS

Donald L. Bott

Science Applications, Inc.
Englewood, Colorado

ABSTRACT

In an inventory model, the distribution of total units demanded can be considered as a compound distribution arising from the distributions of demand occurrence and individual demand size. Three such compound distributions are considered, where the number of demands is Poisson distributed. The demand size distribution will depend on the observed or desired variance-to-mean ratio. An approximation using the gamma distribution is given in terms of the cumulants of the compound distribution for both fixed and stochastic lead times.

I. INTRODUCTION

One of the most fundamental components of a stochastic inventory model is the probability distribution of total units demanded over some period of time. In the widely used models of Hadley and Whitin [6], the general approach is to approximate the total units demanded over a lead time by the Normal distribution. In this discussion, the total number of units demanded over a lead time can be considered as the sum of a random number of demands, each generating a random number of units demanded. This situation is treated through the use of compound distributions.

We shall assume that demands arrive in accordance with a stationary Poisson process with rate δ, i.e., the expected number of demands over a period $(0, T)$ is δT. The choice of distribution for demand size depends on the amount of variability in the historical data on demand size. One measure of relative variability is the variance-to-mean ratio. For example, the Negative Binomial distribution has a variance-to-mean ratio greater than unity for any choice of parameters. If the historical data has a sample variance-to-mean ratio greater than one, we would attempt to fit the demand size data with a Negative Binomial distribution. In simulating inventory models, the Negative Binomial distribution would be a good choice in situations where demand size is expected to vary greatly. Other discrete distributions examined in this paper are the Poisson (variance-to-mean ratio equals unity) and the Binomial (variance-to-mean ratio less than unity).

The properties of the resulting compound distributions are discussed in Johnson and Kotz [7]. One immediate result is that the probability distribution of these distributions presents severe computational difficulties. Accordingly, we present an approximating Gamma distribution which gives good computational results with little effort. Results are given for both fixed and stochastic lead times.

II. SOME PROPERTIES OF COMPOUND DISTRIBUTIONS

In this section, we describe the compound distributions of interest and describe procedures for obtaining their central moments. In each case, demands are assumed to follow a stationary Poisson process with rate δ. For example, suppose that the variance-to-mean ratio of the number of units per demand is less than unity; this suggests the use of the Binomial distribution to describe the number of units per demand, since the variance-to-mean ratio of the Binomial distribution is less than unity for any choice of parameters. The resulting distribution of demand size is given by:

$$Pr[D=k;\, r,\, p]=\binom{r}{k} p^k q^{r-k} \qquad \begin{array}{l} r=1,\, 2,\, \ldots \\ k=0,\, 1,\, \ldots,\, r \\ 0\leq p\leq 1 \\ q=1-p. \end{array} \tag{1}$$

In order to derive the properties of interest, we introduce the probability generating function: suppose that X is a lattice random variable with non-zero probabilities only at non-negative integer values. Define:

$$Pr[X=k]=P_k,\ k=0,\, 1,\, 2,\, \ldots \tag{2}$$

Since $\sum_{k=0}^{\infty} P_k=1$,

$$g(t)=\sum_{k=0}^{\infty} P_k t^k \tag{3}$$

converges for $|t|\leq 1$. The function $g(t)$ is called the *Probability Generating Function* of the distribution given by (1). Johnson and Kotz [7] give the following results which we require:

(4) (i) If $g(t)$ is defined as in (2), then

$$g^{(r)}(1)=\mu_{(r)}$$

where $g^{(r)}(t)$ is the r^{th} derivative of $g(t)$, and $\mu_{(r)}$ is the expected value of X^r, or the r^{th} *descending factorial moment.*

(ii) If $X_1,\ \ldots,\ X_N$ are independent random variables with probability generating functions $g_1(t),\ \ldots,\ g_N(t)$, then the probability generating function of

$$\sum_{j=1}^{N} X_j \text{ is}$$

$$E\left[t^{\sum_{i=1}^{N} X_i}\right].$$

In particular, if $X_1,\ \ldots,\ X_N$ are independent, identically distributed random variables with probability generating function $g(t)$, then the probability generating function of

$$\sum_{j=1}^{N} X_j \text{ is}$$

$$E[(t^x)^N]. \tag{5}$$

We also require the following relationships between the r^{th} descending factorial moment $\mu_{(r)}$ and the r^{th} central moment, μ_r:

$$\mu_2=\mu_{(2)}+\mu_{(1)}-\mu_{(1)}^2 \tag{6}$$

$$\mu_3=\mu_{(3)}+(\mu_{(1)}-1)[\mu_{(1)}(2\mu_{(1)}-1)-3\mu_{(2)}] \tag{7}$$

together with the relationship

$$\mu_1'=\mu_{(1)} \tag{8}$$

where μ_1' is the mean of the distribution.

To return to our example, in a fixed lead time T, the number of demands follows a Poisson distribution with rate δT, where the distribution of demand size is given by (1). It is well known that, for the Binomial,

$$g(t)=(pt+q)^r. \tag{9}$$

Johnson and Kotz show that the probability generating function of the total units demanded is

$$g_N(t)=E[(pt+q)^{rN}]=\exp\{\delta T[(pt+q)^r-1]\} \tag{10}$$

where N has a Poisson distribution with rate δT.

Following the notation of Johnson and Kotz [7], we shall label our compound distributions as follows: if demand size follows the Binomial, the resulting compound is called the *Poisson-Binomial;* if demand size follows the Negative Binomial, the result is called the *Poisson-Pascal;* and if demand size follows the Poisson distribution, the result is called the *Poisson-Poisson* or *Neyman Type A.*

III. THE APPROXIMATING GAMMA DISTRIBUTION

Johnson and Kotz [7] give expressions for the density functions of the above distributions; in general, these expressions are computationally difficult and would represent a significant burden in terms of computer time for large scale applications. We now present a method for approximating the compound distribution by a two-parameter Gamma distribution. The Gamma is chosen because it is "rich," in the sense that the probability density function can assume a variety of shapes depending on the parameters, and because the Gamma distribution has a range from zero to infinity, as does the random variable z which represents total lead time demand.

Following the method of Best and Gipps [2], let K_1, K_2, and K_3 represent the first three central moments of the compound distribution. Define the Gamma density $f(v;\ \alpha,\ \beta)$ as

$$f(v;\alpha,\beta)=\frac{\alpha^\beta}{\Gamma(\beta)}v^{\beta-1}e^{-\alpha v} \qquad v\geq 0 \tag{11}$$

$$\alpha,\ \beta>0.$$

The approximation is obtained by equating the second and third central moments of the compound distribution with those of the Gamma:

$$K_2=\frac{\beta}{\alpha^2} \tag{12}$$

(13) $$K_3=\frac{2\beta}{\alpha^3}.$$

Solving (12) and (13) simultaneously,

(14) $$\alpha=\frac{2K_2}{K_3}$$

and from (12),

(15) $$\beta=K_2\alpha^2$$

$$=\frac{4K_2^3}{K_3^2}.$$

The Gamma distribution with this pair of values for α and β would have a mean of

$$\frac{\beta}{\alpha}=\alpha K_2=\frac{2K_2^2}{K_3}$$

while the corresponding compound distribution will have a mean of K_1, which is

$$\frac{2K_2^2}{K_3}-K_1$$

greater than the mean of the compound distribution. Equating the second and third cumulants thus results in a shift in the origin. Changing the origin does not change cumulants higher than the first, so that we may approximate the cumulative sum of the compound distribution,

$$\sum_{x=0}^{r} Pr[X=x]$$

by the Gamma cumulative distribution

(16) $$\int_h^H f(v;\alpha,\beta)dv$$

where

$$h=\max\left(0,\frac{2K_2^2}{K_3}-K_1-0.5\right)$$

$$H=r+0.5+\frac{2K_2^2}{K_3}-K_1.$$

For the Gamma cumulative distribution,

(17) $$Pr[X\leq r]=\int_h^H f(v;\alpha,\beta)dv.$$

IV. FIXED AND STOCHASTIC LEAD TIMES

Let us put this discussion in the inventory context again. Suppose that demands during a fixed lead time of length T arrive in accordance with a Poisson process with daily rate δ.

Note that (17) depends on the central moments of the compound distribution, through the parameters and limits of integration, and the central moments depend on the lead time T. This is reasonable in the inventory context, and allows us to treat the problem of stochastic lead times directly.

We will assume that the lead time T is a random variable with the Gamma density

$$f(t;\lambda,\eta)=\frac{\lambda^{\eta}}{\Gamma(\eta)}\,t^{\eta-1}e^{-\lambda t} \qquad t\geq 0 \tag{18}$$

$$\lambda,\ \eta>0.$$

Once again the Gamma density is chosen because it is rich and adaptive. We use the properties of conditional expectation to find the probability generating function when lead time is Gamma distributed:

$$g(t)=\int_0^{\infty} g(t|T)\,f(T;\lambda,\eta)\,dT. \tag{19}$$

V. AN EXAMPLE USING THE NEYMAN TYPE A DISTRIBUTION

Recall that the Neyman Type A distribution is the result of a Poisson distribution of arrivals compounded with a Poisson distribution of requisition size. As above, let the distribution of demands during a lead time T have rate δ, and demand size have parameter ρ. Then the distribution of demand size is

$$Pr[D=k]=e^{-\rho}\frac{\rho^k}{k!}$$

and

$$g(t|T)=\exp\left[-\delta T\{1-\exp\left[-\rho(1-t)\right]\}\right]. \tag{20}$$

Using (20), the first three cumulants for a fixed lead time T are

$K_1=\rho\delta T$

$K_2=\rho\delta T(\rho+1)$

$K_3=\rho\delta T(\rho^2+3\rho+1)$.

Now, suppose that the lead time T is a random variable with density given by (18). From (19) and (20),

$$\begin{aligned} g(t)&=\int_0^{\infty} e^{\delta T[e^{\rho(t-1)}-1]}\,\frac{\lambda^{\eta}}{\Gamma(\eta)}\,T^{\eta-1}e^{-\lambda T}dT \\ &=\int_0^{\infty}\frac{\lambda^{\eta}}{\Gamma(\eta)}\,T^{\eta-1}e^{-T\{\lambda+\delta[e^{\rho(t-1)}-1]\}}dT \\ &=\frac{\lambda^{\eta}}{\{\lambda+\delta[e^{\rho(t-1)}-1]\}^{\eta}}\int_0^{\infty}\frac{\{\lambda+\delta[e^{\rho(t-1)}-1]\}^{\eta}}{\Gamma(\eta)}\,T^{\eta-1}\,e^{-T\{\lambda+\delta[e^{\rho(t-1)}-1]\}^{\eta}}dT. \end{aligned}$$

Since the integrand is itself a Gamma density,

$$g(t)=\left[\frac{\lambda}{\lambda+\delta[e^{\rho(t-1)}-1]}\right]^{\eta}. \tag{21}$$

These results, together with (6)–(8) and (17), allow us to compute the approximation to the Neyman Type A distribution function.

VI. SUMMARY OF RESULTS

The results given below summarize the values of K_1, K_2, and K_3 which are required to utilize the approximating Gamma distribution given by (17), where the selection of compound distribution depends on the distribution of demand size.

Neyman Type A

Demand Size Distribution:

$$Pr[D=k;\ \rho]=\frac{e^{-\rho}\rho^k}{k!} \qquad k=0, 1, \ldots$$

Fixed Lead Times:

$K_1=\rho\delta T$

$K_2=\rho\delta T(\rho+1)$

$K_3=\rho\delta T(\rho^2+3\rho+1)$

Stochastic Lead Times:

$K_1=\rho\delta\eta\lambda^{-1}$

$K_2=\rho\delta\eta\lambda^{-2}[\rho(\lambda+\delta)+\lambda]$

$K_3=\rho\delta\eta\lambda^{-3}\{\rho(\lambda+\delta)[\rho(\lambda+2\delta)+3\lambda]+\lambda^2\}$

Poisson-Pascal

Demand Size Distribution:

$$Pr[D=k;\ r, p]=\binom{r+k-1}{k}p^r q^k \qquad r=1, 2, \ldots$$

$$k=0, 1, \ldots$$

$$0\leq p\leq 1$$

Fixed Lead Times:

$K_1=\delta Trqp^{-1}$

$K_2=\delta Trqp^{-2}[q(r+1)+p]$

$K_3=\delta Trqp^{-3}[qr(qr+3)+q+1]$

Stochastic Lead Times:

$K_1=\delta qr\eta(p\lambda)^{-1}$

$K_2=\delta q^2 r\eta(p\lambda)^{-2}[\delta r+\lambda(r+1)]$

$K_3=\delta q^3 r\eta(p\lambda)^{-3}\{\lambda\delta r(rq+1)+q[\delta r+\lambda(r+1)][2\delta r+\lambda(r+2)]\}$

Poisson-Binomial

Demand Size Distribution:

$$Pr[D=k;\, r,\, p]=\binom{r}{k}p^k q^{r-k} \qquad r=1,\, 2,\, \ldots$$
$$0\leq p\leq 1$$
$$k=0,\, 1,\, \ldots,\, r$$

Fixed Lead Times:

$K_1=rp\delta T$

$K_2=rp\delta T(rp+q)$

$K_3=rp\delta T[rp(rp+3q)+q(q-p)]$

Stochastic Lead Times:

$K_1=rp\delta\eta\lambda^{-1}$

$K_2=rp\delta\eta\lambda^{-2}[\lambda q+rp(\lambda+\delta)]$

$K_3=rp\delta\eta\lambda^{-3}\{rp(\lambda+\delta)[\lambda(rp+3q)+2r\delta pq]\}$

VII. A COMPUTATIONAL EXAMPLE—THE NEYMAN TYPE A DISTRIBUTION

Consider the case where demands arrive at rate δ per day, with average demand size of 7.5 and a fixed lead time of 100 days. The cumulative probability distribution of lead time demand, using the Neyman Type A distribution, is given in Table 1 below for various values of δ. The table shows the exact value, Gamma approximation given in (15), and the Normal approximation using the first two moments of the compound distribution.

Table 2 shows the mean of the approximating Gamma distribution when the parameters α and β are computed from (14) and (15), together with the mean of the corresponding compound distribution.

TABLE 1. Distribution of Lead Time Demand

Pr (lead time demand$\leqq k$)

	k	Exact	Gamma	Normal
$\delta = 0.02$	0	.1354	.0182	.0920
	2	.1408	.0642	.1248
	4	.1713	.1219	.1650
	6	.2398	.1890	.2127
	8	.3247	.2621	.2676
	10	.4013	.3381	.3290
	12	.4690	.4138	.3952
	14	.5363	.4868	.4647
	16	.6033	.5551	.5353
	18	.6654	.6174	.6047
	20	.7201	.6730	.6710
	22	.7680	.7217	.7323
	24	.8099	.7636	.7873
	26	.8460	.7992	.8350
	28	.8766	.8290	.8752
	30	.9019	.8536	.9080
	32	.9227	.8738	.9339
	34	.9397	.8900	.9538
	36	.9533	.9031	.9685
	38	.9642	.9134	.9792
	40	.9727	.9217	.9866
	42	.9794	.9281	.9916
	44	.9845	.9331	.9949
	46	.9884	.9370	.9970
	48	.9914	.9399	.9983
	50	.9937	.9422	.9990
$\delta = 0.04$	0	.0184	.0030	.0301
	4	.0281	.0232	.0517
	8	.0724	.0604	.0841
	12	.1300	.1178	.1298
	16	.2087	.1949	.1903
	20	.3009	.2876	.2656
	24	.4008	.3892	.3536
	28	.5022	.4923	.4502
	32	.5985	.5904	.5498
	36	.6855	.6785	.6464
	40	.7605	.7539	.7344
	44	.8225	.8158	.8097
	48	.8718	.8646	.8702
	52	.9096	.9018	.9158
	56	.9378	.9294	.9482
	60	.9581	.9492	.9698
	64	.9724	.9632	.9834
	68	.9822	.9728	.9913
	72	.9887	.9793	.9957
	76	.9930	.9835	.9980
	80	.9957	.9863	.9991

	k	Exact	Gamma	Normal
$\delta = 0.06$	0	.0025	.0006	.0107
	4	.0026	.0047	.0180
	8	.0140	.0133	.0292
	12	.0297	.0288	.0458
	16	.0562	.0541	.0691
	20	.0942	.0910	.1006
	24	.1447	.1406	.1415
	28	.2070	.2024	.1924
	32	.2790	.2745	.2531
	36	.3578	.3539	.3227
	40	.4399	.4370	.3991
	44	.5219	.5199	.4796
	48	.6004	.5993	.5610
	52	.6729	.6725	.6398
	56	.7377	.7376	.7131
	60	.7939	.7938	.7784
	64	.8412	.8409	.8343
	68	.8799	.8793	.8802
	72	.9108	.9099	.9163
	76	.9350	.9337	.9435
	80	.9534	.9518	.9632
	84	.9672	.9653	.9769
	88	.9772	.9752	.9860
	92	.9844	.9824	.9919
	96	.9895	.9874	.9954
	100	.9931	.9910	.9975
	104	.9955	.9934	.9987
$\delta = 0.08$	0	.0003	.0001	.0039
	5	.0003	.0013	.0074
	10	.0040	.0045	.0134
	15	.0110	.0116	.0231
	20	.0250	.0252	.0383
	25	.0488	.0484	.0606
	30	.0852	.0839	.0920
	35	.1354	.1335	.1341
	40	.1994	.1970	.1879
	45	.2752	.2729	.2533
	50	.3595	.3576	.3290
	55	.4480	.4469	.4124
	60	.5363	.5360	.5000
	65	.6204	.6208	.5876
	70	.6971	.6979	.6710
	75	.7643	.7653	.7467
	80	.8211	.8220	.8121
	85	.8675	.8681	.8658
	90	.9041	.9044	.9080
	95	.9321	.9322	.9394
	100	.9530	.9528	.9617
	105	.9682	.9678	.9768
	110	.9789	.9784	.9866
	115	.9863	.9857	.9926
	120	.9912	.9906	.9960
	125	.9945	.9939	.9980

TABLE 1. (Continued)

	k	Exact	Gamma	Normal
$\delta = 0.10$	0	. 0000	. 0000	. 0015
	7	. 0000	. 0005	. 0035
	14	. 0019	. 0023	. 0078
	21	. 0070	. 0076	. 0162
	28	. 0199	. 0204	. 0313
	35	. 0460	. 0459	. 0566
	42	. 0903	. 0894	. 0956
	49	. 1557	. 1542	. 1516
	56	. 2416	. 2400	. 2259
	63	. 3434	. 3421	. 3173
	70	. 4535	. 4529	. 4215
	77	. 5634	. 5636	. 5316
	84	. 6654	. 6661	. 6392
	91	. 7539	. 7549	. 7369
	98	. 8262	. 8271	. 8188
	105	. 8820	. 8826	. 8826
	112	. 9229	. 9232	. 9286
	119	. 9515	. 9515	. 9593
	126	. 9705	. 9704	. 9783
	133	. 9827	. 9824	. 9892
	140	. 9902	. 9899	. 9950
	147	. 9946	. 9943	. 9978
	154	. 9971	. 9969	. 9991

TABLE 2

Gamma and Compound Distribution Means

δ	K_1	$\frac{\beta}{\alpha}$
0. 02	15. 00	27. 17
0. 04	30. 00	54. 35
0. 06	45. 00	81. 33
0. 08	60. 00	108. 70
0. 10	75. 00	135. 87

VIII. CONCLUDING REMARKS

As the example above shows, the approximating Gamma distribution tends to underestimate the cumulative compound distribution. Computational experience has shown that the Gamma approximation tends to have relatively poor agreement with the exact curve when the expected number of demands in a lead time is less than four, and that the Gamma is a better approximation to the Poisson-Binomial and Poisson-Pascal than to the Neyman Type A distribution.

In summary, this paper has attempted to show that the total demand distribution of an inventory model can arise naturally from the distributions of demand occurrence and demand size, and that a useful approximation can be obtained for the resulting compound distribution of total demand.

BIBLIOGRAPHY

[1] Adelson, R. M., "Compound Poisson Distributions," Operational Research Quarterly, *17*(1), 73–75.

[2] Best, D. J. and P. G. Gipps, "An Improved Approximation to the Negative Binomial," Technometrics, *16*(4), 621–624 (1974).

[3] Burgin, T. A., "The Gamma Distribution in Inventory Control," Operational Research Quarterly, *26*(3, i), 507–525 (1975).

[4] Das, C., "Approximate Solution to the (Q_1r) Inventory Model for Gamma Lead Time Demand," Management Science, *22*(9), 1043–1047 (1976).

[5] Galliher, H. P., P. M. Morse, and M. Simond, "Dynamics of Two Classes of Continuous-Review Inventory Systems," Operations Research, *7*(3), 362–384 (1959).

[6] Hadley, G. and T. M. Whitin, *Analysis of Inventory Systems* (Prentice-Hall, Englewood Cliffs, N.J., 1963).

[7] Johnson, N. L. and S. Kotz, *Discrete Distributions* (Houghton Mifflin, Boston, 1969).

[8] Johnson, N. L. and S. Kotz, *Continuous Univariate Distributions–1* (Houghton Mifflin, Boston, 1970).

[9] Kendall, M. G. and A. Stuart, *The Advanced Theory of Statistics, I* (Hafner, New York, 1969).

[10] Silver, E. A., "Some Ideas Related to the Inventory Control of Items having Erratic Demand Patterns," Canadian Operational Research Journal, *8*(2), 87–100 (1970).

[11] Stegun, I. A., and M. Abramowitz, *Handbook of Mathematical Functions*, NBS Applied Mathematics Series 55 (U.S. Government Printing Office, Washington, D.C., 1964).

PROBABILITY DENSITY FUNCTION OF A STOCHASTIC LINEAR PROGRAMMING PROBLEM

Wei Shen Hsia

University of Alabama
University, Alabama

ABSTRACT

A stochastic linear programming problem under the "wait-and-see" situation is studied. After the conditions for the interchange of the order of integration and differentiation are surveyed, an explicit form of the probability density function of the stochastic linear programming problem is found. An example is also given to illustrate the result.

1. INTRODUCTION

Consider the following linear programming problem

$$\begin{array}{lll} & \text{maximize} & c^t x \\ (1) & \text{subject to} & Ax=\mathbf{b} \\ & & x\geq 0 \end{array}$$

where c, x are $n\times 1$ vectors, A is an $m\times n$ matrix, $\mathbf{b}$ is an $m\times 1$ random vector on (R^m, β, P) with a known absolutely continuous distribution function.

There are two one-stage stochastic programming situations, the "here-and-now" and the "wait-and-see" situations. The "here-and-now" situation was suggested by Dantzig [7] and Beale [3] independently: A decision x must first be made, then the random event $\mathbf{b}$ observed, and finally the "inaccuracies" in the first decision corrected by an activity vector. G. Tintner [13] was the first to consider the "wait-and-see" situation: The random event $\mathbf{b}$ is observed first and then a nonstochastic linear programming problem is solved on the basis of this observed $\mathbf{b}$. Then $z(\mathbf{b})=\max\{c^t x|Ax=\mathbf{b}, x\geq 0\}$ is a random variable and $E[z(\mathbf{b})]$ is called the solution value of the "wait-and-see" situation. Related results and surveys of these fields can be found in numerous papers, e.g., [2, 5, 9, 10, 11, 12, 14, 15, 16, 17, 18].

It is the purpose of this paper to study the probability density function (p.d.f.) of the random variable $z(\mathbf{b})$. Therefore, from now on, only the "wait-and-see" situation of (1) is considered. If the random vector $\mathbf{b}$ is discrete (finite), it is only necessary to solve a finite number of linear programs to get the p.d.f. of the random variable $z(\mathbf{b})$. However, if $\mathbf{b}$ is continuous, finding the

p.d f. of $z(\mathbf{b})$ explicitly may be extremely difficult, because an infinite number of programming problems may be involved.

Based on results in [8], which are stated in Section 2, the p.d.f. of $z(\mathbf{b})$ could be found. However, a big obstacle when finding the p.d.f. of $z(\mathbf{b})$ is the inerchange of the order of integration and differentiation. In Section 3, conditions to justify the interchange of the order of integration and differentiation are given. The solution value of an example is then given in Section 4.

2. PROBABILITY DISTRIBUTION FUNCTION

Assume the matrix $A=[A_1, \ldots, A_n]$ has rank $m \leq n$. A sequence $S=\{i_1, \ldots, i_m\}$ is called a basic sequence of A if the matrix $[A_{i_1}, \ldots, A_{i_m}]$ is nonsingular. Let T' denote the set of all basic sequences of A.

For each $S \epsilon T'$, denote the corresponding basic variables, non-basic variables, basic cost vector, and non-singular basic matrix of A by x_S, $x_{S'}$, c_S, and B_S, respectively.

For a given observed $\mathbf{b}$, a basic sequence S is said to be an optimal basic sequence if

$$c_S B_S^{-1} A_j - c_j \geq 0, \text{ for all } j=1, \ldots, n, \tag{2}$$

and

$$B_S^{-1}\mathbf{b} \geq 0. \tag{3}$$

Note that (2) is independent of the random vector $\mathbf{b}$. Therefore, the basic sequence which does not satisfy (2) can be discarded. Let T denote $\{S | S \epsilon T'$ and (2) is satisfied$\}$. Bereanu [4] states that the probability space β of $\mathbf{b}$ is partitioned by $\{M_S | S \epsilon T\} \cup \{N\}$ where

$$M_S=\{\mathbf{b} | B_S^{-1}\mathbf{b} \geq 0\}, \quad N=\{\mathbf{b} | z(\mathbf{b}) \text{ does not exist}\}.$$

Furthermore, $Pr(\{\mathbf{b} | (B_S^{-1}\mathbf{b})_j = 0$ for at least one $j \epsilon S\}) = 0$ if the joint p.d.f. of $\mathbf{b}$ is continuous. Also $Pr(\{M_{S_1} \cap M_{S_2}\}) = 0$ for $S_1 \neq S_2$.

For simplicity, assume $Pr(N) = 0$. Then for each given real number t,

$$\begin{aligned} Pr(\{z(\mathbf{b}) \leq t\}) &= \sum_{S \epsilon T} Pr(\{z(\mathbf{b}) \leq t\} \cap M_S) \\ &= \sum_{S \epsilon T} Pr(\{\mathbf{b} | c_S{}^t (B_S^{-1}\mathbf{b} \leq t;\ B_S^{-1}\mathbf{b} \geq 0\}) \\ &= \sum_{S \epsilon T} \int_{\{\mathbf{b} | c_S{}^t (B_S^{-1}\mathbf{b}) \leq t;} \cdots \int_{B_S^{-1}\mathbf{b} \geq 0\}} f(\mathbf{b}) db_1 \ldots db_m \\ &= \sum_{S \epsilon T} F_S(t), \end{aligned} \tag{4}$$

where f is the joint p.d.f. of $\mathbf{b}$, and

$$F_S(t) = \int_{\{\mathbf{b} | c_S{}^t (B_S^{-1}\mathbf{b}) \leq t;} \cdots \int_{B_S^{-1}\mathbf{b} \geq 0\}} f(\mathbf{b}) db_1 \ldots db_m.$$

The limits of the integration involved in (4) are very difficult to obtain if $m \geq 3$. Charnes, Cooper, and Thompson [6] suggested a method to overcome this difficulty by transformation which is stated as follows.

Let $S \epsilon T$, define a new $m \times 1$ random vector $\mathbf{r}$ by $\mathbf{r}=B_S^{-1}\mathbf{b}$. Then

$$F_S(t)=\int_{\{\mathbf{r}|c_S'\mathbf{r}\leq t,\mathbf{r}\geq 0\}} \cdots \int f(B_S\mathbf{r})|J_S|dr_1 \ldots dr_m$$

where $J_S=\det(B)$ is the Jacobian of the transformation.

If there is an $i \epsilon S$ for which $c_i \leq 0$, r_i is allowed to vary from zero to infinity. If the r_j's are reordered so that all of those r_j's whose corresponding coefficients in the vector c_S are less than or equal to zero come at the end of the $\mathbf{r}$ vector. The index of the whole system is also reordered accordingly. Let q be the number r_j's remaining. The inequalities $c_S'r \leq t$ then becomes

$$\sum_{j=1}^{q} c_j r_j \leq t - \sum_{j=q+1}^{m} c_j r_j.$$

Therefore, r_j can vary from 0 to

$$h_j(\mathbf{r})=\frac{1}{c_j}\left[t-\sum_{k=j+1}^{m} c_k r_k\right], \text{ for } j=1, \ldots, q.$$

Thus,

$$F_S(t)=\int_0^\infty \cdots \int_0^\infty \int_0^{h_q(\mathbf{r})} \cdots \int_0^{h_1(\mathbf{r})} f(B_S\mathbf{r})|\det(B)|dr_1 \ldots dr_q \ldots dr_m.$$

And, the probability distribution function of $z(\mathbf{b})$ is

$$Pr\{z(\mathbf{b})\leq t\}=\sum_{S \epsilon T}\int_0^\infty \cdots \int_0^\infty \int_0^{h_q(\mathbf{r})} \cdots \int_0^{h_1(\mathbf{r})} f(B_S\mathbf{r})|\det(B_S)|dr_1 \ldots dr_m. \tag{5}$$

The solution value of (1) is defined by $E[z(\mathbf{b})]$. Therefore, the p.d.f. of $z(\mathbf{b})$ must be found. If the p.d.f. of $z(\mathbf{b})$ is found by taking the derivative of (2) with respect to t, it is necessary to involve the interchange of the order of integration and differentiation. Conditions which will insure that the order of integration and differentiation be interchangeable are discussed in the next section.

3. PROBABILITY DENSITY FUNCTION

Several lemmas and theorems are considered first.

LEMMA 1: Suppose

$$\int_0^{a_m} \cdots \int_0^{a_1} f(r_1, \ldots, r_m; t)\, dr_1 \ldots dr_m$$

converges pointwise to g on $[c, d]$. If the partial derivative

$$\frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}$$

is continuous on a compact set $D=[0, a_1]\times \ldots \times[0, a_m]\times[c, d]$ then

$$g'(t)=\int_0^{a_m} \cdots \int_0^{a_1} \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}\, dr_1, \ldots dr_m.$$

PROOF: Since

$$\frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}$$

is continuous on the compact set D, for each positive number ϵ, there exists a $\delta>0$, depending on ϵ only, such that if $(r_1, \ldots, r_m; t)$, $(r'_1, \ldots, r'_m; t) \epsilon D$, and $|(r_1, \ldots, r_m; t)-(r'_1, \ldots, r'_m; t')|<\delta$, it follows

$$\left|\frac{\partial f(r_1, \ldots, r_m; t)}{\partial t} - \frac{\partial f(r'_1, \ldots, r'_m; t')}{\partial t}\right| < \min\left\{\frac{\epsilon}{a_i^m} \middle| i=1, \ldots, m\right\}.$$

Let $(r_1, \ldots, r_m; t)$, $(r_1, \ldots, r_m; t+h) \epsilon D$ with $0<|h|<\delta$, then by the Mean Value Theorem,

$$\begin{aligned}
&\left|\int_0^{a_m} \cdots \int_0^{a_1} \frac{f(r_1, \ldots, r_m; t+h)-f(r_1, \ldots, r_m; t)}{h} - \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}\, dr_1 \ldots dr_m\right| \\
&= \left|\int_0^{a_m} \cdots \int_0^{a_1} \frac{\partial f(r_1, \ldots, r_m; t+\theta h)}{\partial t} - \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}\, dr_1, \ldots dr_m\right| \\
&\leq \int_0^{a_m} \cdots \int_0^{a_1} \left|\frac{\partial f(r_1, \ldots, r_m; t+\theta h)}{\partial t} - \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}\right| dr_1, \ldots dr_m \\
&\leq \min\left\{\frac{\epsilon}{a_i^m} \mid i=1, \ldots, m\right\} \times a_1 a_2 \ldots a_m \\
&\leq \epsilon,
\end{aligned}$$

where θ is a number in (0, 1). Therefore,

$$g'(t) = \int_0^{a_m} \cdots \int_0^{a_1} \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}\, dr_1, \ldots dr_m. \qquad \text{Q.E.D}$$

LEMMA 2: Assume that each term of the sequence $\{g_m\}$ is a real valued function having a finite derivative at each point in (a, b). Assume that for at least one point x_0 in (a, b) the sequence $\{g_m(x_0)\}$ converges. Assume further that there exists a function h such that $\{g_m'\}$ converges uniformly to h on (a, b). Then

(a) there exists a function g such that $\{g_m\}$ converges uniformly to g on (a, b), and
(b) for each $t \epsilon (a, b)$, the derivative $g'(t)$ exists and equals $h(t)$.

PROOF: See [1].

THEOREM 1: Suppose

$$\int_0^{\infty} \cdots \int_0^{\infty} f(r_1, \ldots, r_m; t)\, dr_1 \ldots dr_m$$

converges pointwise to g on $[c, d]$. If the partial derivative

$$\frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}$$

is continuous on $[0, \infty] \times \ldots \times [0, \infty] \times [c, d]$, and if integral

$$\int_0^{\infty} \cdots \int_0^{\infty} \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t}\, dr_1 \ldots dr_m$$

converges uniformly on $[c, d]$, then $g'(t)$ exists for each $t \epsilon (c, d)$, and

$$g'(t) = \int_0^\infty \cdots \int_0^\infty \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t} dr_1 \ldots dr_m.$$

PROOF: For each positive integer n, define

$$g_n(t) = \int_0^n \cdots \int_0^n f(r_1, \ldots, r_m; t) dr_1 \ldots dr_m.$$

By Lemma 1, for each $t \epsilon (c, d)$,

$$g'_n(t) = \int_0^n \cdots \int_0^n \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t} dr_1 \ldots dr_m.$$

Therefore, the sequence of functions $\{g_n\}$ converges pointwise to g on $[c, d]$, and $\{g'_n\}$ converges to

$$\int_0^\infty \cdots \int_0^\infty \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t} dr_1 \ldots dr_m.$$

By Lemma 2, it follows

$$g'(t) = \int_0^\infty \cdots \int_0^\infty \frac{\partial f(r_1, \ldots, r_m; t)}{\partial t} dr_1 \ldots dr_m \qquad \text{Q.E.D.}$$

Next, consider the derivative of another type of function.

The following lemma is cited from [1].

LEMMA 3: Assuma that $f(r; t)$ and $\frac{\partial f(r; t)}{\partial t}$ are continuous on $[a, b] \times [c, d]$.

Let h be a function having a finite derivative h' on $[c, d]$ and assume $h(t) \epsilon [a, b]$, for every $t \epsilon [c, d]$. Define

$$g(t) = \int_0^{h(t)} f(r; t) dr, \text{ for } t \epsilon [c, d].$$

Then $g'(t)$ exists for every $t \epsilon (c, d)$, and is given by

$$g'(t) = \int_0^{h(t)} \frac{\partial f(r; t)}{\partial t} dr + f(h(t), t)\, h'(t).$$

For any given $\mathbf{r} = (r_1, \ldots, r_m)'$, $t \epsilon [c, d]$, and for $i = 2, \ldots, q+1$, define

$$g_i(r_i, \ldots, r_m; t) = \int_0^{h_{i-1}(\mathbf{r}; t)} \cdots \int_0^{h_1(\mathbf{r}; t)} f(r_1, \ldots, r_m)\, dr_1 \ldots dr_{i-1},$$

where $h_j(\mathbf{r}; t)$, $j = 1, \ldots, q$, are defined in Section 2, and $a_i \leq h_j(\mathbf{r}; t) \leq b_i$. Define

$$g_1(r_1, \ldots, r_m; t) = f(r_1, \ldots, r_m),$$

where f is the joint p.d.f. of $\mathbf{b}$, then, for $i = 2, \ldots, q+1$,

$$g_i(r_i, \ldots, r_m; t) = \int_0^{h_{i-1}(\mathbf{r}; t)} g_{i-1}(r_{i-1}, \ldots, r_m; t) dr_{i-1}.$$

The following lemma can be obtained after a slight modification is made on Lemma 3.

LEMMA 4: Assume $\int_0^{h_{i-1}(\mathbf{r};\,t)} g_{i-1}(r_{i-1}, \ldots, r_m;\, t)dr_{i-1}$ coverges pointwise to

$$g_i(r_i, \ldots, r_m;\, t) \text{ on } [a_i, b_i]\times \ldots \times[a_m, b_m]\times[c, d].$$

Assume that g_{i-1} and $\frac{\partial g_{i-1}(r_{i-1}, \ldots, r_m;\, t)}{\partial t}$ are continuous on $[a_{i-1}, b_{i-1}]\times \ldots \times[a_m, b_m]\times[c, d]$. If $\frac{\partial h(\mathbf{r};\, t)}{\partial t}$ exists and is finite, then

$$\frac{\partial g_i(r_i, \ldots, r_m;\, t)}{\partial t}=f_{1,\,i}(r_i, \ldots, r_m;\, t)+f_{2,\,i}(r_i, \ldots, r_m;\, t)$$

where

$$f_{1,\,i}(r_i, \ldots, r_m;\, t)=\int_0^{h_{i-1}(\mathbf{r};\,t)} \frac{\partial g_{i-1}(r_{i-1}, \ldots, r_m;\, t)}{\partial t}\, dr_{i-1}$$

$$f_{2,\,i}(r_i, \ldots, r_m;\, t)=q_{i-1}(h_{i-1}(\mathbf{r};\, t), r_i, \ldots, r_m;\, t)\, \frac{\partial h_{i-1}(\mathbf{r};\, t)}{\partial t}$$

The partial derivative of $g_q(r_{q+1}, \ldots, r_m;\, t)$, with respect to t can be obtained by q applications of Lemma 4,

$$\frac{\partial g_q(r_{q+1}, \ldots, r_m;\, t)}{\partial t}=\sum_{i=1}^{q} H_i(r_{q+1}, \ldots, r_m;\, t), \tag{6}$$

where, for $i=1, \ldots q-1$,

$$H_i(r_{q+1}, \ldots, r_m;\, t)=\int_0^{h_q(\mathbf{r};\,t)} \cdots \int_0^{h_{i+1}(\mathbf{r};\,t)} f_{2,\,i+1}(r_{i+1}, \ldots, r_m;\, t)\, dr_{i+1} \ldots dr_q\ .$$

and

$$H_q(r_{q+1}, \ldots, r_m;\, t)=f_{2,\,q+1}(r_{q+1}, \ldots, r_m;\, t).$$

Combine Theorem 1 and (6), the p.d.f. of $z(\mathbf{b})$ contributed by a given basic sequence $S\epsilon T$, $f_S(t)$, is

$$f_S(t)=F'_S(t)=\int_0^{\infty} \cdots \int_0^{\infty} \frac{\partial g_{q+1}(r_{q+1}, \ldots, r_m;\, t)}{\partial t}\, dr_{q+1} \ldots dr_m$$

$$=\sum_{i=1}^{q} \int_0^{\infty} \cdots \int_0^{\infty} H_i(r_{q+1}, \ldots, r_m;\, t)\, dr_{q+1} \ldots dr_m. \tag{7}$$

Finally, the p.d.f. of $z(\mathbf{b})$ is $\sum_{S\,\epsilon\,T} f_S(t)$.

4. EXAMPLE

The following example is the one used in [8].

$$\text{maximize} \qquad 2x_1+3x_2$$

Subject to

$$\begin{aligned} x_1+x_2+x_3 &= b_1 \\ x_1+2x_2+x_4 &= b_2 \\ x_i &\geq 0, \quad i=1, 2, 3, 4, \end{aligned} \tag{8}$$

where b_1, b_2 are independent random variables, both having exponential densities with parameters 1 and 2, respectively. The joint p.d.f. of b_1 and b_2 is

$$f(b_1, b_2)\begin{cases}2e^{-(b_1+2b_2)} & \text{for } b_1, b_2>0\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\{1, 2\}$, $\{1, 3\}$, $\{2, 4\}$ are the only three basic sequences which satisfy (2).

For $S=\{1, 2\}$: $x_S=(x_1, x_2)'$, $B=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$,

$Br=(r_1+r_2, r_1+2r_2)'$, $f(Br)=2e^{-(3r_1+5r_2)}$, $|J|=1$.

The contribution of $\{1, 2\}$ to the distribution function of $z(\mathbf{b})$ is

$$F_{\{1,2\}}(t)\int_0^{t/3}\int_0^{(t/2)-(3r_2/2)}2e^{-(3r_1+5r_2)}dr_1dr_2, \text{ for } t\geq 0.$$

By (7), the contribution of $\{1, 2\}$ to the p.d.f. of $z(\mathbf{b})$ is

$$f_{\{1,2\}}(t)=(1/2)\int_0^{t/3}2e^{-(7t/2+r_2/2)}dr_2$$

$$=-2e^{-5t/3}+2e^{3t/2} \qquad \text{for } t\geq 0.$$

Similarly, the contribution of $\{1, 3\}$, and $\{2, 4\}$ to the p.d.f. of $z(\mathbf{b})$ are

$$f_{\{1,3\}}(t)=(1/2)\int_0^{\infty}2e^{-(3t/2+r_2)}dr_2=e^{-3t/2}, \text{ for } t\geq 0,$$

and

$$f_{\{2,4\}}(t)=(1/3)\int_0^{\infty}2e^{-(5t/3+2r_2)}dr_2=(1/3)e^{-5t/3}, \text{ for } t\geq 0.$$

The solution value of the stochastic program (8) then is

$$E[z(\mathbf{b})]=3\int_0^{\infty}te^{-3t/2}dt-(5/3)\int_0^{\infty}te^{-5t/3}dt$$

$$=11/15$$

with standard deviation 7/5.

5. ACKNOWLEDGMENT

The author gratefully acknowledges the support of this research by the University of Alabama under Project No. 891.

REFERENCES

[1] Apostol, T. M., "Mathematical Analysis," (Addison-Wesley Publishing Company, Inc., 1957).

[2] Babbar, M. M., "Distributions of Solutions of a Set of Linear Equation (With An Application To Linear Programming)," Journal of the American Statistical Association, *50*, 854–869 (1955).

[3] Beale, E. M. L., "On minimizing a Convex Function Subject To Linear Inequalities," Journal of the Royal Statistical Society, B., *XVII*, 173–184, (1955).

[4] Bereanu, B., "On Stochastic Linear Programming I: Distribution Problems: A Single Random Variable," Revue Roumaine de Mathématiques Pures et Appliquées *VIII*, (Bucharest), 863–697 (1963).

[5] Charnes, A. and W. W. Cooper, "Chance-Constrained Programming," Management Science, *6*, 73–79 (1959).

[6] Charnes, A., W. W. Cooper, and G. L. Thompson, "Critical Path Analysis Via Chance Constrained, and Stochastic Programming," Operations Research, *12*, 460–470 (1964).

[7] Dantzig, G. B., "Linear Programming Under Uncertainty," Management Science, *1*, 197–206 (1955).

[8] Ewbank, J. B., B. L. Foote, and H. T. Kumin, "A Method For The Solution of The Distribution Problem of Stochastic Linear Programming," SIAM Journal of Applied Mathematics, *26*, 225–238 (1974).

[9] Freund, R. J., "The Introduction of Risk Into a Programming Model," Econometrica, *24*, 253–263 (1956).

[10] Madansky, A. "Inequalities For Stochastic Linear Programming Problems," Managemen Science, *6*, 197–204 (1960).

[11] Mangasarian, O. L., "Nonlinear Programming Problem With Stochastic Objective Functions," Management Science, *12*, 353–359 (1964).

[12] Talacko, J. V., "On Stochastic Linear Inequalities," Trahajos de Estadistica, *10*, 89–112 (1959)

[13] Tintner, G., "Stochastic Linear Programming With Applications to Agricultural Economics," second Symposium in Linear Programming (ed. H. Antosiewicz), Washington, D.C., 197–227 (1955).

[14] Tintner, G., "A Note On Stochastic Linear Programming," Econometrica, *28*, 490–495 (1960).

[15] Vajda, S., "Inequalities In Stochastic Linear Programming," Bulletin of the International Statistics Institute, *36*, 357–363 (1958).

[16] Wagner, H. M., "On The Distribution of Solutions In Linear Programming Problems," Journal of the American Statistical Association, *53*, 161–163 (1958).

[17] Wets, R. J. B., "Stochastic Programs With Fixed Recourse: The Equivalent Deterministic Program," SIAM Review, *16*, 309–339 (1974).

[18] Ziemba, W. T., "Stochastic Programs With Simple Recourse," Technical Report 72–75, (Operations Research House, Stanford University, 1972).

SEQUENTIAL TESTS FOR HYPOTHESES CONCERNING THE PRODUCT OF POISSON PARAMETERS WITH APPLICATIONS TO PROBLEMS IN RELIABILITY AND BIOMETRY*

B. Harris and A. P. Soms

Mathematics Research Center
University of Wisconsin—Madison
Madison, Wisconsin

ABSTRACT

Sequential tests for the product of Poisson parameters based on the generalized incomplete modified Bessel (g.i.m.B.) distributions are given. Applications to reliability and biometry are indicated.

. INTRODUCTION AND SUMMARY

In this paper, we obtain a sequential test for the product of Poisson parameters. Such a test ıay be interpreted as an approximate test for the product of binomial parameters, a problem with ıany practical applications. To illustrate this, we describe some situations in which the product f binomial parameters is a natural parameter of interest.

In reliability theory, a system is said to be a parallel system if a system failure occurs only with ıe failure of every component. Then if p_i is the probability of failure of the ith component, $i=1, 2, \ldots, k$; the probability of the failure of the system is

$$\rho = \prod_{i=1}^{k} p_i,$$

ssuming that the components are stochastically independent.

Similarly, assume that k independent screening methods are employed to detect the presence f a disease. Then if p_i, $i=1, 2, \ldots, k$, is the probability that the ith screening method will not etect the disease, it follows that ρ is the probability that an individual with the disease escapes etection. Specifically, tuberculosis provides an appropriate illustration, since both skin tests and -ray screening methods can be employed and their performance may be assumed to be indepen- ent.

*Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

Thus, it is quite natural to seek tests of hypotheses for ρ. In this paper, we will proceed by first approximating the binomial distribution by the Poisson distribution. That is, if X_i is binomially distributed with parameter (N_i, p_i), then we will regard X_i as having the Poisson distribution with parameter $\lambda_i=N_ip_i$, $i=1, 2, \ldots, k$. Note that the approximation of the binomial distribution by the Poisson distribution is in conformity with the suggested applications, since it is natural to assume that the p_i's are "small." If we denote

$$\prod_{i=1}^{k} \lambda_i$$

by θ, then it follows that a test of any hypothesis concerning θ may be regarded as an approximate test for

$$\rho \prod_{i=1}^{k} N_i.$$

Thus rejection of the hypothesis $\theta>\theta_0$ is interpreted as rejection of the hypothesis $\rho>\rho_0$, where

$$\rho_0=\theta_0 \Big/ \prod_{i=1}^{k} N_i.$$

A fixed sample size test for θ has been previously given in B. Harris [1]. In this brief note, we show how the techniques of that paper may be employed to obtain a sequential test. We proceed by first describing the generalized incomplete modified Bessel distributions in Section 2. These have the property that they depend on the individual λ_i's only through θ and thus they are suited for tests of hypotheses concerning θ. The sequential tests are constructed in Section 3, and Section 4 is devoted to some numerical illustrations. The tests described in this paper are carried out very easily, involving only very simple computations. These computations may be even further reduced in some instances by use of the tables of Harris and Soms [3].

2. THE GENERALIZED INCOMPLETE MODIFIED BESSEL DISTRIBUTIONS

In this section we derive the generalized incomplete modified Bessel distributions and note in particular certain properties which make them particularly suited for the problem at hand.

Let X_i, $i=1, 2, \ldots, k$ be independent Poisson random variables with $E(X_i)=\lambda_i$, $\lambda_i>0$. Then the joint distribution of $X_1, X_2, \ldots, X_k$ is given by

$$p(\tilde{x}; \tilde{\lambda})=e^{-\sum_{i=1}^{k}\lambda_i} \prod_{i=1}^{k} (\lambda_i^{x_i}/x_i!), \quad x_i=0, 1, \ldots, i=1, 2, \ldots, k; \tag{2.1}$$

where $\tilde{x}=(x_1, x_2, \ldots, x_k)$ and $\tilde{\lambda}=(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let $U_1=X_1$ and let $U_i=X_i-X_1$, $i=2, \ldots, k$. Then the joint distribution of $U_1, U_2, \ldots, U_k$ is given by

$$p_1(\tilde{u}; \lambda)=e^{-\sum_{i=1}^{k}\lambda_i} (\lambda_1^{u_1}/u_1!) \prod_{i=2}^{k} \lambda_i^{u_i+u_1}/(u_i+u_1)!, \tag{2.2}$$

where $u_1=0, 1, 2, \ldots$ and $u_i=-u_1, -u_1+1, -u_1+2, \ldots, i=2, 3, \ldots, k$. Thus, the conditional distribution of U_1 given $U_2=u_2, U_3=u_3, \ldots, U_k=u_k$ is given by

2.3) $$p_\theta(u_1|u_2, \ldots, u_k)=\theta^{u_1}/u_1!(u_2+u_1)!(u_3+u_1)! \ldots (u_k+u_1)!h(u_2, \ldots, u_k; \theta),$$

where

2.4) $$h(u_2, \ldots, u_k; \theta)=\sum_{u_1}\theta^{u_1}/u_1!(u_2+u_1)!(u_3+u_1)! \ldots (u_k+u_1)!$$

nd $\max(0, \max_{2\leq i\leq k}(-u_i))\leq u_1<\infty$, which is also the range of the sum in (2.4). With no loss of enerality, we can take u_1 to be the minimal observation among $x_1, x_2, \ldots, x_k$. This permits us to implify (2.3) by restricting the range of u_1 to $u_1\geq 0$, a convention which we will adopt for the alance of this paper.

In particular for $k=2$, we have

2.5) $$P_\theta\{U_1\leq t|U_2=u_2\}=I_{u_2}(t, 2\sqrt{\theta})/I_{u_2}(2\sqrt{\theta}),$$

here

2.6) $$I_\nu(t, \kappa)=(\kappa/2)^\nu\sum_{k=0}^{t}\frac{(\kappa^2/4)^k}{k!(\nu+k)!}$$

nd $I_\nu(\infty, \kappa)=I_\nu(\kappa)$ is the modified Bessel function of order ν. Hence, these distributions are amed the generalized incomplete modified Bessel distributions, which we henceforth abbreviate g.i.m. B. distributions. Many properties of these distributions are given in Harris [1] and Harris nd Soms [2].

We now proceed to the construction of the sequential tests.

THE SEQUENTIAL TESTS

Let X_{ij}, $i=1, 2, \ldots, k$, $j=1, 2, \ldots$ be independent Poisson random variables with $E(X_{ij})=\lambda_i$, $=1, 2, \ldots, k$ for all j. Then, for each j, the conditional distribution of U_{1j} given $U_{2j}=u_{2j}, \ldots$, $_{kj}=u_{kj}$ is the g.i.m.B. distribution with parameters θ, $u_{2j}, u_{3j}, \ldots, u_{kj}$.

We proceed by employing the Wald sequential probability ratio test to construct a test of $_0: \theta=\theta_0$ against the alternative $H_1: \theta=\theta_1$. To obtain a test of approximate size α and power $1-\beta$, e set $b=\log(\beta/(1-\alpha))$ and $a=\log((1-\beta)/\alpha)$. The reader is referred to Wald [4], pp. 44–48 for details ncerning the relationships between the actual size and actual power of the test and the nominal lues α and $1-\beta$. For the g.i.m.B. distribution (2.3), the logarithm of the likelihood ratio is given by

3.1) $$Z_{u_2, \ldots, u_k; \theta_0, \theta_1}(U_1)=\log\{h(u_2, \ldots, u_k; \theta_0)/h(u_2, \ldots, u_k; \theta_1)\}+U_1\log(\theta_1/\theta_0).$$

Thus, the sequential probability ratio test is specified by

.2) $$\begin{cases}\text{reject } H_0 \text{ when } \Sigma Z_j\geq a,\\ \text{accept } H_0 \text{ when } \Sigma Z_j\leq b,\\ \text{if } b<\Sigma Z_j<a, \text{ continue sampling,}\end{cases}$$

here $Z_j=Z_{u_2, \ldots, u_k; \theta_0, \theta_1}(U_{1j})$, $j=1,2, \ldots$. For $k=2$, $0\leq u_2\leq 5$ and $k=3$, $0\leq u_2, u_3\leq 3$, the quired values of $h_\theta(u_2)$, $h_\theta(u_2,u_3)$ are tabulated in [3] for $0\leq\theta\leq 100$, where they are denoted by H.

Some remarks concerning the applicability of the sequential probability ratio test are needed, since (2.3) is conditioned on $u_2, \ldots, u_k$. On the sample space determined by the random variables $X_{1j}, X_{2j}, \ldots, X_{kj}, j=1,2, \ldots$; the Z_j's are independent identically distributed random variables; hence the proof that the sequential probability ratio test terminates with probability one (Wald [4], pp. 157–158) applies. Also, the fact that the relations between (a, b) and (α, β) apply conditionally insures that they will also hold unconditionally. Unfortunately, the distribution of Z_j appears to be quite complicated and consequently various quantities of interest, such as the average sampling number, are as yet unknown to us.

To illustrate the procedures described above, we now present some numerical illustrations.

4. NUMERICAL ILLUSTRATIONS

While the procedure described here is of interest in itself, this technique is more likely to be employed as an approximate test for the product of binomial parameters. The examples given below are constructed for this purpose. In these illustrations, logarithms to base 10 have been employed. We will test $H_0:\rho_0=.001$ against $H_1:\rho_1=.01$ where $\rho=p_1p_2$, the product of two binomial parameters. Let $\alpha=\beta=.05$; then $a=1.27875=-b$. The data given below in Tables 1–3 is to be interpreted as data from two binomial populations with parameters (N_1,p_1), (N_2, p_2) and p_i, $i=1,2$ is to be regarded as the probability of failure of the ith component in a parallel system with two components. Thus $\rho=p_1p_2$ is the probability of a system failure. We have set $N_1=N_2=10$. To illustrate the procedure, three sets of artificial data have been constructed. In the first (Table 1), we have data from the binomial populations (10, .01) and (10, .1). In the second (Table 2), data from the binomial populations (10, .1), (10, .1) is provided, and in the third (Table 3), data from the binomial populations (10, .05), (10, .1) is given. Thus in the first H_0 is true, in the second H_1 is true and in the third, neither hypothesis is true. The examples follow.

TABLE 1. Data from the binomial distributions with $N_1=N_2=10$, $p_1=.01$, $p_2=.1$ used to test the hypothesis $H_0:\rho=p_1p_2=.001$ against the alternative $\rho=.01$.

j	$X_{1j}=U_{1j}$	X_{2j}	$U_{2j}=X_{2j}-X_{1j}$	Z_j	$\sum_{i=1}^{j} Z_i$
1	0	2	2	−.1248	−.1248
2	0	0	0	−.3155	−.4403
3	0	2	2	−.1248	−.5651
4	0	0	0	−.3155	−.8806
5	0	2	2	−.1248	−1.0054
6	0	2	2	−.1248	−1.1302
7	0	0	0	−.3155	*−1.4457
8	0	3	3		
9	0	0	0		
10	0	1	1		
11	0	0	0		
12	0	1	1		
13	0	1	1		
14	0	1	1		
15	0	1	1		

After the seventh observation, sampling is stopped and H_0 is accepted.

TABLE 2. Data from the binomial distributions with $N_1=N_2=10$, $p_1=p_2=.1$ used to test the hypothesis $H_0:\rho=p_1p_2=.001$ against the alternative $\rho=.01$.

j	$X_{1j}=U_{1j}$	X_{2j}	$U_{2j}=X_{2j}-X_{1j}$	Z_j	$\sum_{i=1}^{j} Z_i$
1	1	1	0	.6845	.6845
2	1	0	−1	−.1800	.5045
3	0	1	1	−.1800	.3245
4	2	0	−2	−.1248	.1997
5	0	0	0	−.3155	−.1158
6	2	2	0	1.6845	*1.5687
7	1	1	0		
8	3	1	−2		
9	0	1	1		
10	2	1	−1		
11	1	1	0		
12	1	0	−1		
13	0	1	1		
14	0	0	0		
15	0	1	1		

After the sixth observation sampling is stopped and H_0 is rejected.

TABLE 3. Data from the binomial distributions with $N_1=N_2=10$, $p_1=.05$, $p_2=.1$ used to test the hypothesis $H_0:\rho=p_1p_2=.001$ against the alternative $\rho=.01$.

j	$X_{1j}=U_{1j}$	X_{2j}	$U_{2j}=X_{2j}-X_{1j}$	Z_j	$\sum_{i=1}^{j} Z_i$
1	0	1	1	−.1800	−.1800
2	1	1	0	.6845	.5045
3	2	1	−1	.8200	*1.3245
4	1	2	1		
5	0	1	1		
6	0	1	1		
7	0	1	1		
8	0	1	1		
9	0	2	2		
10	0	2	2		
11	0	1	1		
12	1	1	0		
13	0	2	2		
14	0	1	1		
15	0	0	0		

After the third observation sampling is terminate and H_0 is rejected.

In the above examples we have provided additional observations which were not actually ɜed in the sequential test. This was done to provide the reader with a better picture of the data that he could convince himself that the data given do in fact look like random samples from the ated populations.

REFERENCES

[1] Harris, B., "Hypothesis testing and confidence intervals for products and quotients of Poisson parameters with applications to reliability," Journal of the American Statistical Association, *66*, 609–613 (1971).

[2] Harris, B. and A. P. Soms, "Properties of the generalized incomplete modified Bessel distributions with applications to reliability theory," Journal of the American Statistical Association, *69*, 259–263 (1974).

[3] Harris, B. and A. P. Soms, "Tables of the two-factor and three-factor generalized incomplete modified Bessel distributions," *Selected Tables in Mathematical Statistics*, *3*, 1–109, American Mathematical Society, Providence, 1975.

[4] Wald, A., *Sequential Analysis*, (John Wiley and Sons, Inc., New York, 1947).

RELATION BETWEEN CONTINUOUS AND DISCRETE TIME MARKOVIAN DECISION PROBLEMS

Prasadarao Kakumanu

Polytechnic Institute of New York
Brooklyn, New York

ABSTRACT

An associated discrete (continuous) time Markovian decision problem for any given continuous (discrete) time Markovian decision model is formulated. A relationship between the continuous and associated discrete time discounted returns is obtained. This result is used to show the existence of a deterministic stationary policy that optimizes both discounted return functions. It is also proved that the same policy optimizes the average expected return for both continuous and the associated discrete time processes. The results obtained in this paper can be used to solve the continuous time Markovian decision problem by using the discrete time algorithms or vice versa.

. INTRODUCTION

In this paper we investigate the relationship between continuous and discrete time Markovian ecision Processes. In the continuous case the process can be observed at any time, while, in the iscrete case, it is observed periodically. In both cases, once the process is observed and classified ıto one of the possible states, an action from a set of possible actions is taken and as a result, the rocess receives a return and moves to a new state according to a known law of motion. Hence, the Iarkovian decision process is completely specified if the following four objects are given.

(a). The set of possible states called the state set denoted by S, assumed countable.

(b). The set of possible actions called action set denoted by A, assumed finite.

(c). The law of motion describes how the process moves from state to state. In the continuous ıse this is called the transition rate matrix and is denoted by $Q(a)=\{q_{ij}(a),\ i, j\epsilon S\},\ a\epsilon A$. The approximate probability of going from state i to state j in time δt is given by $q_{ij}(a)\ \delta t,\ i\neq j$, and $-q_{ii}(a)\delta t)$, for each $a\epsilon A$. In the discrete time model it is the one-step transition probability ıatrix and is denoted by $P(a)=\{p_{ij}(a);\ i, j\epsilon S\},\ a\epsilon A$, where $p_{ij}(a)$ is the probability of going from ate i to state j in one period.

(d). Return to the process is a function of both the state of the process and the action taken. hat is, if the state of the process is $i\epsilon S$ and an action $a\epsilon A$ is taken then a return $r(i, a)$ is obined for both the continuous and the associated discrete processes. The return $r(i, a)$ is assumed) be uniformly bounded in i and a.

Policy

A policy is a rule that specifies action (s) at each point in time. In this paper a policy will be denoted by π (t) or simply by π. A policy π might depend on the entire history of the process up to time t, and specify a probability distribution on the action set A instead of choosing a single action. In this paper we assume that a policy π depends only on the present state of the process. Such a policy can be specified by a set of functions $\{d_{ia}(t)\}$, with the property

$$\Sigma_a d_{ia}(t)=1 \text{ and } d_{ia}(t)\geq 0,\ i\epsilon S,\ a\epsilon A, \text{ and } t\geq 0. \tag{1.1}$$

The function $d_{ia}(t)$ may be interpreted as the probability of taking action "a" at time t, given the state of the process is $i\epsilon S$. Any policy π satisfying these conditions is called a *randomized policy*. A policy π is said to be a *measurable policy* if $d_{ia}(t)$ is measurable in t for all $i\epsilon S$ and $a\epsilon A$. We call a measurable randomized policy a *Markov policy*. A Markov policy π is defined if and only if the set of measurable functions $\{d_{ia}(t)\}$ satisfying (1.1) is known. A Markov policy π is called a *stationary policy* if

$$d_{ia}(t)=d_{ia}\ i\epsilon S,\ a\epsilon A, \text{ and } t\geq 0. \tag{1.2}$$

That is, for each $i\epsilon S$, a stationary policy specifies a probability distribution on A, which is independent of t. A stationary policy is called a deterministic policy if for each $i\epsilon S$:

$$d_{ia}=1, \text{ or } 0, \text{ for all } i\epsilon S \text{ and } a\epsilon A. \tag{1.3}$$

The time parameter in the discrete case is a non-negative integer. The policy $\delta=\delta(n)$ is also defined by a set of numbers $\{y_{ia}(n)\}$ which satisfies (1.1) for each n. If these sets of numbers satisfy (1.2) the policy δ is called stationary; and if they also satisfy (1.3), it is called a deterministic stationary policy. The set of stationary policies and the set of deterministic stationary policies are identical in the discrete and the continuous time models.

Markov Process

Let π be any Markov policy specified by a set of functions $\{d_{ia}(t)\}$ satisfying (1.1). When this policy π is used, the transition rate from state i to state j is given by:

$$q_{ij}(t,\pi)=\Sigma_a q_{ij}(a)d_{ia}(t),\ i,\ j\epsilon S, \text{ and } t\geq 0. \tag{1.4}$$

The transition rate matrix $Q(t,\pi)$ whose $(i,j)th$ element is given by (1.4) is called bounded if its elements satisfy:

$$\begin{cases}\Sigma_j q_{ij}(t,\pi)=0;\ q_{ij}(t,\pi)\geq 0;\ i\neq j,\ i,\ j\epsilon S\\ |q_{ii}(t,\pi)|<M, \text{ where } M \text{ is a finite positive number.}\end{cases} \tag{1.5}$$

If $Q(t,\pi)$ is bounded, then there exists [2] a unique transition probability matrix

$$F(s,t,\pi)=\{f_{ij}(s,t,\pi);\ i,\ j\epsilon S\},$$

where $f_{ij}(s,t,\pi)$; $i,\ j\epsilon S$ and $t>s$ is the probability of going from state i to state j in time $(t-s)$. It was further shown that there exists a Markov process $X(t,\pi)$ such that for any $t\geq s$ and π

1.6) $$f_{ij}(s, t, \pi)=P\{X(t, \pi)=j|X(s, \pi)=i\};\ i, j\epsilon S.$$

In general, $X(t, \pi)$ is a non-stationary Markov process but, if the policy π is stationary, $X(t, \pi)$ s a time-homogeneous Markov process. When the policy π is stationary we use the notation $Q(\pi)$ nd $F(t, \pi)$, instead of $Q(t, \pi)$ and $F(s, t, \pi)$ respectively.

ssociated Discrete Time Markov Process

We first define a one-step transition probability matrix $P(a)$, corresponding to a given bounded ransition rate matrix $Q(a)$, and later an associated discrete time Markovian decision process.

et

$$\lambda=\sup_{i, a}||q_{ii}(a)||.$$

t is obvious that λ is a finite and strictly positive number. Define

1.7) $$p_{ij}(a)=\delta_{ij}+\lambda^{-1}q_{ij}(a),\ i, j\epsilon S,\ a\epsilon A,$$

here $\delta_{ij}=1$ if $i=j$, and 0 if $i\neq j$. The one step transition probability matrix is given by

1.8) $$P(a)=\{p_{ij}(a);\ i, j\epsilon S\},\ a\epsilon A.$$

et δ be any Markov policy given by $\{y_{ia}(n)\}$. When the policy δ is used, the probability of going o state j in one step, given that the state of the process is i at time n, is given by:

1.9) $$p_{ij}(n, \delta)=\Sigma_a p_{ij}(\delta)y_{ia}(n),\ i, j\epsilon S,\ n\geq 0.$$

f the policy δ is stationary, $y_{ia}(n)$ is independent of n which implies $p_{ij}(n, \pi)$ is independent of n. When the policy δ is stationary, the one-step transition probability matrix is denoted by:

$$P(\delta)=\{p_{ij}(\delta);\ i, j\epsilon S\}.$$

Given a $P(\delta)$, it can be shown ([1], p. 7) the existence of a stationary discrete time Markov rocess $Y_n(\delta)$ whose one-step transition probabilities are given by:

$$p_{ij}(\delta)=P\{Y_n(\delta)=j||Y_{n-1}(\delta)=i\},\ i, j\epsilon S, n\geq 0.$$

We assume the same initial probability distribution for both the processes $X(t, \pi)$, and $Y_n(\delta)$. We call $X(t, \pi)$ a continuous time Markovian decision process, and $Y_n(\delta)$ an associated discrete ime Markovian decision process of $X(t, \pi)$.

I. DISCOUNTED CRITERION FUNCTIONS

In this section we define discounted return functions for both continuous and associated dis- rete time Markovian decision processes and show they are linerally related. This result will be sed to prove that the same policy optimizes both criterion functions and that the optimal return or one process can be obtained from the optimal return for the other.

Let π be an arbitrary Markov policy. When the policy π is used the expected rate of return out of state i is given by:

$$r(i, t, \pi)=\Sigma_a r(i, a)d_{ia}(t),\ i\epsilon S, \text{ and } t\geq 0. \tag{2.1}$$

The rate of return during the interval $[s, t)$ may be written as $\int_s^t r(i, u, \pi)\, du$. The total expected discounted return over the infinite horizon for the continuous time Markovian decision process is defined to be:

$$\Psi(i, \alpha, \pi)=\int_0^\infty e^{-\alpha t}\Sigma_j f_{ij}(0, t, \pi)\, r(j, t, \pi)\, dt\ i\epsilon S, \tag{2.2}$$

where $\alpha>0$ is called the discount factor.

Now we will define a discounted criterion function for the associated process. Let δ be any arbitrary Markov policy and let β be any real number such that $0<\beta<1$. When the policy δ is used, the expected one period return out of state i is given by:

$$r(i, n, \delta)=\Sigma_a r(i, a)d_{ia}(n),\ i\epsilon S, \text{ and } n\geq 0. \tag{2.3}$$

The total expected discounted return for the discrete time Markovian decision process when the policy $\delta(n)$ is used is given by:

$$V(i, \beta, \delta)=\Sigma_n \beta^n \Sigma_j p_{ij}^{(n)}(\delta) r(j, n, \delta),\ i\epsilon S, \tag{2.4}$$

where β is called the discount factor, and $p_{ij}^{(n)}(\delta)$ is the n-step transition probability from state i to state j when the policy $\delta(n)$ is used. Let

$$\Psi(i\ \alpha)=\sup_\pi \Psi(i, \alpha, \pi), \text{ and } V(i, \beta)=\sup_\delta V(i, \beta, \delta),\ i,\epsilon S.$$

A policy π^* is called α-discounted optimal policy if $\Psi(i, \alpha, \pi^*)=\Psi(i, \alpha)$, $i\epsilon S$. And a policy δ^* is called β-discounted optimal policy if $V(i, \beta, \delta^*)=V(i, \beta)$, $i\epsilon S$.

For the continuous time problem, the existence of a α-discounted optimal deterministic stationary policy in the class of deterministic Markov policies is proved in [3], and this result is extended to all Markov policies in [5]. Similar results for the discrete time problem are also known and given in [6], chapter 6. In the following two theorems we take $\beta=\lambda/(\alpha+\lambda)$, and show that $\Psi(i, \alpha, \pi)$ is proportional to $V(i, \beta, \pi)$, $i\epsilon S$, and that the same deterministic stationary policy maximizes both return functions.

THEOREM 2.1. For any stationary policy π

$$(\alpha+\lambda)\Psi(i, \alpha, \pi)=V(i, \beta, \pi),\ i\epsilon S.$$

PROOF: For any stationary policy π the α-discounted return $\Psi(i, \alpha, \pi)$, uniquely satisfies the functional equation ([3], p. 921)

$$\alpha\Psi(i, \alpha, \pi)=r(i, \pi)+\Sigma_j q_{ij}(\pi)\Psi(j, \alpha, \pi),\ j\epsilon S.$$

Writing $q_{ij}(\pi)$ in terms of $p_{ij}(\pi)$ by using (1.7) we obtain

$$\alpha\Psi(i, \alpha, \pi)=r(i, \pi)+\Sigma_j\lambda[p_{ij}(\pi)-\delta_{ij}]\Psi(j, \alpha, \pi),\ j\epsilon S.$$

fter some simplification and rearranging of terms we arrive at

$$(\alpha+\lambda)\Psi(i, \alpha, \pi)=r(i,\pi)+\lambda\Sigma_j p_{ij}(\pi)\Psi(j, \alpha, \pi),\ j\epsilon S.$$

et $\tilde{r}(i, \pi)=\frac{r(i, \pi)}{\alpha+\lambda}$, and since $\beta=\lambda/(\alpha+\lambda)$, the above equation may be written as

$$\Psi(i, \alpha, \pi)=\tilde{r}(i, \pi)+\beta\Sigma_j p_{ij}(\pi)\Psi(j, \alpha, \pi),\ i\epsilon S.$$

$$=\tilde{r}(i, \pi)+\beta\Sigma_j p_{ij}(\pi)\tilde{r}(j, \pi)$$

$$+\beta^2\Sigma_j\Sigma_k p_{ij}(\pi)p_{jk}(\pi)\Psi(k, \alpha, \pi),\ i\epsilon S.$$

sing Chapman-Kolmogorov equations ([1], p. 9) we obtain

$$\Psi(i, \alpha, \pi)=\tilde{r}(i, \pi)+\beta\Sigma_j p_{ij}(\pi)\tilde{r}(j, \pi)+\beta^2\Sigma_k p_{ik}^{(2)}(\pi)\Psi(k, \alpha, \pi).$$

epeating this infinite times we arrive at

$$\Psi(i, \alpha, \pi)=\Sigma_n\beta^n\Sigma_j p_{ij}^{(n)}(\pi)\tilde{r}(j, \pi),\ i\epsilon S$$

$$=\frac{1}{\alpha+\lambda}\Sigma_n\beta^n\Sigma_j p_{ij}^{(n)}(\pi)r(j, \pi),\ i\epsilon S$$

$$=\frac{1}{a+\lambda}V(i, \beta, \pi),\ i\epsilon S$$

hich was to be proved.

THEOREM 2.2. A policy π^* is an α-discounted optimal deterministic stationary policy for ie continuous time problem if and only if it is a β-discounted optimal deterministic stationary olicy for the associated discrete time problem.

PROOF. In Theorem 2.1 we showed that for any stationary policy π, $\Psi(i, \alpha, \pi)$ is proportional $V(i, \beta, \pi)$ for each $i\epsilon S$, and the constant of proportionality is independent of π. In particular, iese results are true for any deterministic stationary policy. Hence, if a deterministic stationary olicy π^* optimizes $\Psi(i, \alpha, \pi)$ among all deterministic stationary policies for any $i\epsilon S$, it also optimizes (i, β, π), for that i, and vice versa.

If π^* is optimal for the discounted return $V(i, \beta, \pi)$, $i\epsilon S$, over all deterministic stationary olicies, it was shown in [6], p. 121, that it is also optimal over all Markov policies. A similar result as discussed for the discounted return $\Psi(i, \alpha, \pi)$, $i\epsilon S$ in [3], p. 925. This proves that the same eterministic stationary policy π^* optimizes both the discounted returns over all Markov policies.

Using the above two theorems, a solution for a continuous time Markovian decision problem n be obtained by solving the associated discrete time problem. This might be solved by using ther successive approximations, or Howard's policy iteration or linear programming methods. hese results are also helpful to parametrize the discount factor α and obtain the optimal deterinistic stationary policies for different intervals of α.

I. AVERAGE EXPECTED RETURN

In this section, we will show that for any stationary policy the expected average return for ie continuous time decision process is equal to the expected average return for the associated

discrete time decision process. Hence, if the discrete problem is solved, the solution for the continuous problem is automatically available.

The long run average expected returns for the continuous and discrete time Markovian decision processes are respectively defined to be:

$$\Phi(i, \pi)=\lim_{T\to\infty} T^{-1} \int_0^T \Sigma_j f_{ij}(0, t, \pi) r(j, t, \pi) dt, \; i \epsilon S \tag{3.1}$$

$$U(i, \delta)=\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} \Sigma_j p_{ij}^{(n)}(\delta) r(j, n, \delta), \; i \epsilon S \tag{3.2}$$

where π and δ are arbitrary Markov policies respectively for the continuous and the associated discrete processes. If the limit in (3.1) or (3.2) does not exist, we take it to be $-\infty$. A policy π^* is called average optimal if $\Phi(i, \pi^*) \geq \Phi(i, \pi)$ for all Markov π, and δ^* is average optimal if $U(i, \delta^*) \geq U(i, \delta)$ for all Markov δ, for each $i \epsilon S$.

Let k be a fixed state in S. To prove the results in this part we make the assumption

$$q_{ik}(a) > 0, \; i \epsilon S, \; a \epsilon A. \tag{3.3}$$

Under this assumption, the existence of an average optimal deterministic stationary policy for the return function (3.1) was shown in [4], p. 214. Using (3.3) and (1.7), it can be easily shown that

$$p_{ik}(a) > 0, \; i \epsilon S, \; a \epsilon A. \tag{3.4}$$

Ross ([6], Chapter 6), under the assumption (3.4), showed the existence of an average optimal deterministic stationary policy for the criterion function (3.2). Using the results in [4] and [6] we will show that the same deterministic policy optimizes both return functions (3.1) and (3.2).

THEOREM 3.1. Under the assumption (3.3) for an arbitrary deterministic stationary policy π, the long run average expected return for the continuous time and for its associated discrete time decision process have the same value.

PROOF: Let π be any deterministic stationary policy. Under the assumption (3.3), the existence of a finite number $h(\pi)$ and a set of bounded number $V_j(\pi)$, $j \epsilon S$ were shown ([4], p. 215) satisfying the relation:

$$h(\pi)=r(i, \pi)+\Sigma_j q_{ij}(\pi) V_j(\pi), \; i \epsilon S, \tag{3.5}$$

Substituting the values for $q_{ij}(\pi)$ from (1.7), and after some simplification and rearranging of terms, we arrive at:

$$\begin{aligned} h(\pi)+\lambda V_i(\pi) &= r(i, \pi)+\Sigma_j p_{ij}(\pi) \lambda V_j(\pi), \; i \epsilon S \\ \lambda V_i(\pi) &= r(i, \pi)+\Sigma_j p_{ij}(\pi) \lambda V_j(\pi)-h(\pi), \; i \epsilon S \\ &= r(i, \pi)+\Sigma_j p_{ij}(\pi)[r(i, \pi)+\Sigma_k p_{jk}(\pi) \lambda V_k(\pi)-h(\pi)]-h(\pi), \; i \epsilon S. \end{aligned}$$

Using Chapman-Kolmogorov forward equations ([1], p. 9) this may be written as

$$\lambda V_i(\pi)=r(i, \pi)+\Sigma_j p_{ij}(\pi) r(j, \pi)+\Sigma_j p_{ij}^{(2)}(\pi) \lambda V_j(\pi)-2h(\pi), \; i \epsilon S.$$

epeating this N times and rearranging the terms we arrive at

$$Nh(\pi)=\sum_{n=0}^{N-1} p_{ij}^{(n)}(\pi)r(j,\pi)+\Sigma_j[p_{ij}^{(N)}(\pi)-\delta_{ij}](\lambda V_j(\pi)),\ i\epsilon S.$$

ow dividing on both sides by N and taking limit as $N\to\infty$ we obtain

$$h(\pi)=\lim_{N\to\infty} N^{-1}\sum_{n=0}^{N-1}\Sigma_j p_{ij}^{(n)}(\pi)r(j,\pi),\ i\epsilon S;$$

.6)

$$=U(i,\pi),\ i\epsilon S.$$

ultiplying both sides of (3.5) by $f_i f_{ii}(t,\pi)$, and summing over all i and using the Kolomogorov rward differential equation ([1], p. 245) we arrive at:

$$h(\pi)=\Sigma_i f_{ii}(t,\pi)r(i,\pi)+\Sigma_j \frac{d}{dt} f_{ij}(t,\pi)V_j(\pi),\qquad l\epsilon S.$$

tegrating between 0 and T, and dividing by T we obtain:

$$h(\pi)=\frac{1}{T}\int_0^T \Sigma_i f_{ii}(t,\pi)r(i,\pi)dt+\frac{1}{T}\int_0^T \Sigma_j \frac{d}{dt} f_{ij}(t,\pi)V_j(\pi)\ dt,\ l\epsilon S.$$

ow taking limit as $T\to\infty$ ([6], p. 216) we obtain

.7)

$$h(\pi)=\lim_{T\to\infty}\int_0^T \Sigma_j f_{ij}(t,\pi)r(j,\pi)dt,\qquad l\epsilon S.$$

$$=\phi(l,\pi)\qquad l\epsilon S.$$

mparing (3.6) and (3.7) we have proved the theorem.

From (3.6) and (3.7) it is clear that the average expected return is independent of the starting ate for continuous as well as the associated discrete time decision processes.

THEOREM 3.2. If the assumption (3.3) holds, then a deterministic stationary policy π is timal for $\Phi(i,\pi)$, $i\epsilon S$, if, and only if, it is optimal with respect to $U(i,\pi)$, $i\epsilon S$.

PROOF: From the previous theorem we have for any deterministic stationary policy π

.8)

$$\Phi(i,\pi)=U(i,\pi),\qquad i\epsilon S.$$

om this it is obvious if π^* optimizes one criterion function it also optimize the other one.

If a deterministic stationary policy π^* is optimal among all deterministic stationary policies is also optimal among all Markov policies. This was proved for the continuous case in [5], p. 12, d for the discrete case in [2], p. 1550. From these results and (3.8), we conclude that the same terministic policy π^* optimizes both the continuous and the associated discrete average return iterion functions (3.1) and (3.2).

Again using the methodology developed in this section, a continuous time problem may be ved by using discrete time algorithms such as Howard's policy iteration, or linear programming.

IV. ASSOCIATED CONTINUOUS TIME MARKOVIAN DECISION PROCESSES

In this section we will formulate the discrete Markovian decision problem as a continuous time Markovian decision problem. We also indicate the proofs of the results which are similar to those given in Sections II and III. Let the state set, action set, law of motion, and return function be as defined in Section I. The law of motion is given by a one-step transition probability matrix

$$P(a)=\{p_{ij}(a);\ i, j\epsilon S\} \qquad a\epsilon A. \tag{4.1}$$

The corresponding transition rate matrix is defined by

$$Q(a)=\{q_{ij}(a);\ i, j\epsilon S\}, \qquad a\epsilon A$$

where

$$q_{ij}(a)=p_{ij}(a)-\delta_{ij}, \qquad i, j\epsilon S,\ a\epsilon A. \tag{4.2}$$

It is clear that $Q(a)$ is a bounded transition rate matrix. Hence, an associated continuous time Markovian decision process $X(t, \delta)$ corresponding to the discrete time Markovian decision process $Y_n(\delta)$ exists. Here we note that $P(a)$ is given and $Q(a)$ is obtained by (4.2).

Let β and α be the discount factors, corresponding to the discrete and continuous time problems such that $0<\beta<1$, and $\alpha>0$. The discrete and continuous time discounted returns are as defined in Section II. These discounted returns satisfy the relation

$$\beta V(i, \beta, \delta)=\Psi\left(i, \frac{\beta}{1-\beta}, \delta\right), \qquad i\epsilon S, \tag{4.3}$$

where δ is any stationary policy. The proof of this result is similar to that of Theorem 2.1, and can be shown using (2.4), (4.2) and the results in [3]. Now by Theorem 2.2, the same deterministic stationary policy optimizes the discrete and the associated continuous time discounted returns among all Markov policies.

It can also be shown that a single deterministic stationary policy optimizes both the discrete and the associated continuous time average expected returns. To show this, we assume the one-step transition probabilities satisfy

$$p_{ik}(a)>0, \qquad i\epsilon S,\ a\epsilon A, \tag{4.4}$$

where k is some fixed state in S. From this and (4.2) we have

$$q_{ik}(a)>0, \qquad i\epsilon S,\ a\epsilon A. \tag{4.5}$$

Under the assumption (4.4) it can be shown ([6], p. 149) that there exist a bounded set of number $\{h(\delta),\ V_i(\delta);\ i\epsilon S\}$ satisfying the relation

$$h(\delta)+V_i(\delta)=r(i, \delta)+\Sigma_j p_{ij}(\delta)V_j(\delta), \qquad i\epsilon S, \tag{4.6}$$

for any stationary policy δ. Now using (4.2) and (4.6) as in Theorem 3.1, we can show

$$U(i, \delta)=\phi(i, \delta) \qquad i\epsilon S.$$

Using Theorem 3.2 we conclude that the same deterministic stationary policy optimizes both $U(i, \delta)$ and $\phi(i, \delta)$, among all Markov policies.

ACKNOWLEDGMENT

The author would like to thank the referee for his helpful comments.

REFERENCES

1] Chung, K. L., *Markov Chains with Stationary Transition Probabilities*, 2nd ed. (Springer-Verlag, Berlin, 1967).

2] Derman, C., "Denumerable State Markovian Decision Processes—Average Cost Criterion," Annals of Mathematical Statistics, *37*, 1545–1553 (1966).

3] Kakumanu, P., "Continuously Discounted Markov Decision Model with Countable State and Action Space," Annals of Mathematical Statistics, *42*, 919–926 (1971).

4] Kakumanu, P., "Non discounted Continuous Time Markovian Decision Process with Countable State Space," SIAM Journal of Control, *10*, 210–220 (1972).

5] Kakumanu, P., "Continuous Time Recurrent Markovian Decision Processes—Average Return Criterion," Journal of Mathematical Analysis and Application, *52*, 173–188 (1975).

6] Ross, S. M., *Applied Probability Models with Optimization Applications*, (Holden-Day, San Francisco, Calif., 1970).

A NONCONVEX MAX-MIN PROBLEM*

James E. Falk
The George Washington University
Washington, D.C.

Karla Hoffman
National Bureau of Standards
Washington, D.C.

ABSTRACT

An algorithm designed to solve a large class of nonconvex max-min problems is described. Its usefulness and applicability is demonstrated by solving an extension of a recently introduced model which optimally allocates strategic weapon systems. The extended model is shown to be equivalent to a nonconvex mathematical program with an infinite number of constraints, and hence is not solvable by conventional procedures. An example is worked out in detail to illustrate the algorithm.

1. INTRODUCTION

In [4], Shere and Wingate give the details of a model which they designed to optimally, allocate strategic weapon systems. Mathematically, their model is a max-min problem with independent constraints on the players. The inside problem is a convex program with a single complicating constraint.

In this paper, we extend the above model by allowing a more general objective function and more general constraints. The resulting model is characterized by having a nonconvex inside problem and cannot be solved by extending the technique suggested in [4].

The main purpose of this paper is to demonstrate a global solution procedure for general "separable" max-min problems. The technique is actually a composition of two optimization algorithms, one for nonconvex, separable programs, and the other for problems with an infinite number of constraints. It applies to any max-min problem of the form

$$\max_{x \in X} \min_{y \in Y} F(x, y) \Big\} \text{ Problem P,}$$

where $F(x,y) = \sum_i \sum_j F_{ij}(x_i, y_j)$ and where the sets X and Y are described by separable inequalities, e.g.,

$$X\colon\ f(x) = \sum_i f_i(x_i) \leq 0$$

*This work was sponsored by the Office of Naval Research under Program in Logistics Contract N00014-67-A-0214, Task 0001, Project NR 347 020.

$$l \leq x \leq L$$

$$Y: \; g(y) = \sum_j g_j(y_j) \leq 0.$$

$$m \leq y \leq M$$

Note that the functional form of F results from the assumption that $F(\cdot, y)$ is separable as a function of x for each $y \in Y$ and $F(x, \cdot)$ is separable in y for each $x \in X$.

To insure that the above problem has a solution, we shall require

(a) that l, L, m, M are finite vectors,

(b) that each component of the vector values functions f and g and the functions $F(x, \cdot)$ and $-F(\cdot, y)$ are at least lower semi-continuous.

With these assumptions, Problem P must have at least one globally optimal point (x^*, y^*).

Section 2 contains a brief description of the Shere-Wingate model and indicates a number of extensions that increase its applicability. In particular, we allow their search effort to apply to more than one type of system and allow for changes in the unit procurement costs of a system which would be reasonable if quantity discounts are available.

Section 3 contains the details of the algorithm and Section 4 contains small example of the solution of an extended Shere-Wingate problem.

2. HISTORICAL DEVELOPMENT OF THE PROBLEM AND EXTENSIONS

A brief description of the Shere-Wingate model follows for readers unfamiliar with the original paper. Since our emphasis in the present paper is to show that difficult max-min problems can be solved, we do not concentrate on the details of this particular model. Interested readers are referred to [4].

Shere and Wingate discuss both missile and bomber systems. The missile systems are divided into two basic types. The first class is comprised of systems that are difficult to locate but easy to destroy once located (e.g., POLARIS). These systems are called "percentage vulnerable" systems. If y_i is the amount of search effort employed by an attacker in dollars, then the survivability of attack given an original force of x_i dollars of missile system i is measured by the surviving fraction of weapons

$$x_i e^{-a_i y_i}$$

where a_i is the vulnerability of the system.

The second class consists of systems which are easy to locate but difficult to destroy (e.g. MINUTEMEN). These systems are called "numerically vulnerable" systems. The amount of weapons surviving a barrage attack is $x_i e^{-a_i y_i / x_i}$ where, again, y_i is the attacker's level of effort and x_i is the retaliator's level of effort, both measured in dollars.

Thus Wingate and Shere conclude that for a mix of M percentage vulnerable systems and $N-M$ numerically vulnerable systems, the surviving "value" of the mix is

$$F(x, y) = \sum_{i=1}^{M} v_i x_i e^{-a_i y_i} + \sum_{i=M+1}^{N} v_i x_i e^{-a_i y_i / x_i}$$

where the v_i's are values assigned to the systems and measured, say, in megatonnage per dollar

The above model does not consider the possibility that the search effort y_i might be able to locate more than one type of system. This paper has extended the model to include this possibility. If there are M percentage vulnerable systems and $P-M$ numerically vulnerable systems, and if the attacker has N search systems which attempt to locate any of the first class of systems and $Q-N$ search systems which have the capability to attack any of the second class of missile systems, then the surviving value of an attack would be

$$F(x, y)=\sum_{i=1}^{M} v_i x_i \left(\sum_{j=1}^{N} e^{-a_i y_i}\right)+\sum_{i=M+1}^{P} v_i x_i \left(\sum_{j=N+1}^{Q} e^{-a_i y_i/x_i}\right).$$

Modifications to include combinations of attack systems capable of searching for only certain missile systems of either class would not involve additional computational difficulties.

Another extension of the above model allows changes in vulnerability as the attacker's force increases. This change would require a_i to be a function of y_i, and is easily handled by the method to be described.

In addition to the above modifications in missile systems, our model has also included bomber systems using the formula for the fraction of surviving bombers as $(1-b_i \arctan a_i y_i)$ where y_i is the number of reentry vehicles attacking the bomber bases, and a_i and b_i are positive numbers depending on the number of bases, reaction time and density of attack. Thus, when including bomber systems, the surviving value of an attack becomes

(1)

$$F(x, y)=\left\{\sum_{i=1}^{M} v_i x_i \left(\sum_{j=1}^{N} e^{-a_i y_i}\right)\right\}+\left\{\sum_{i=M+1}^{P} v_i x_i \left(\sum_{j=N+1}^{Q} e^{-a_i y_i/x_i}\right)\right\}+\left\{\sum_{i=P+1}^{P+R} v_i x_i (1-b_i \arctan a_{i-P+Q} y_{i-P+Q})\right\}.$$

It should be noted that when bomber systems are added to the model, the function $F(x, \cdot)$ is no longer convex in y.

Applying a conservative strategy, the objective of this model is to determine the optimal allocations x^* and y^* such that $F(x^*, y^*)=\max_x \min_y F(x, y)$ subject to the fiscal, political and arms limitation constraints imposed on the model.

If R_x is the total funds available to the retaliator over a specified time frame and if R_y is the funds available to the attacker, then Wingate and Shere considered the cost for the retaliator to be

$$\sum_{i=1}^{P+R} \{o_i(x_i+\hat{x}_i)/2+p_i \max (0, x_i-\hat{x}_i)+q_i (\operatorname{sgn} x_i)\} \leq R_x$$

where o_i, p_i, and q_i, respectively, denote the operating, procurement and buy-in costs per weapon; x_i is the number of weapons at the end of the time period and $\hat{x}_i$ is the existing or initial number of weapons. Note that $q_i=0$ if either $\hat{x}_i>0$ or $x_i=0$; otherwise it is positive. This paper extends the cost constraint developed by Wingate and Shere to allow not only jumps in the cost curve due to an initial procurement cost but also shifts in the slope of the cost curve, since it is reasonable to have the cost per item decrease due to economics of scale. Figure 1 illustrates the types of cost curves that may be considered.

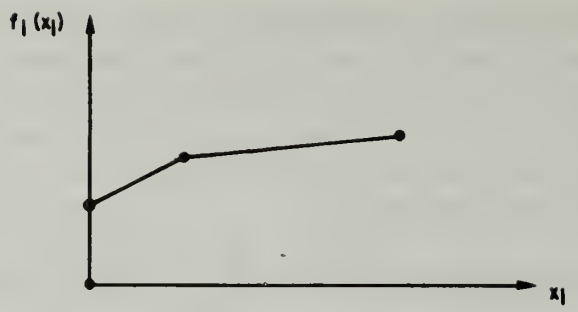


FIGURE 1. A Typical Cost Curve

The cost constraint for the retaliator therefore has the form

$$\sum_{i=1}^{P+R} f_i(x_i) \leq R_x. \tag{2}$$

A similar constraint for the attacker is

$$\sum_{i=1}^{Q+R} g_i(y_i) \leq R_y, \tag{3}$$

where g_i has a form similar to (2).

In addition to these budgetary constraints, there are arms limitations constraints imposed by international treaties. The constraints relating to these treaties given in the Wingate and Shere article are included here, name y

$$\sum_{i=1}^{N} s_i x_i \leq S \tag{4}$$

$$\sum_{i \in M_1} s_i x_i \leq S_1 \tag{5}$$

$$\sum_{i \in M_2} s_i x_i \leq S_2 \tag{6}$$

when the three constraints relate to the total number of launchers, sea-based launchers (M_1) an land-based launchers (M_2), respectively.

Another extension of their model included in this paper is the imposition of an additional con straint on the attacker. Since the arms limitations agreements are so loose as to never be a bindin constraint on the attacker, we have added the more likely political constraint which states that th attacker can only use a proportion of his total force on any one attack, i.e.,

$$\sum_{i=1}^{Q+R} y_i \leq cY,$$

where Y is the total armament storage of the attacker and c is some positive number between ze and one.

Any of the sample political constraints presented in the Wingate and Shere article can also b included in this model. These constraints together with the constraints on all the x and y variabl make up the total constraints of the model.

To solve their model, Wingate and Shere used the fact that the inner problem

$$\min_{y \epsilon Y} F(x, y)$$

was a convex programming problem with one complicating constraint. Our extensions have made the inner problem nonconvex and have introduced more than one complicating constraint so that the algorithm employed is no longer applicable. The next section of this paper will describe how such problems can be solved.

3. THE ALGORITHM

Any max-min problem of the form of Problem P

$$\max_{x \epsilon X} \min_{y \epsilon Y} F(x, y)$$

is equivalent to an optimization problem of the form

$$\max_{x \epsilon X} x_0$$

$$\text{subject to } F(x, y) \geq x_0 \text{ for all } y \epsilon Y.$$

Note that there is a constraint for each point $y \epsilon Y$ so that, in effect, there are an infinite number of constraints on the problem.

Recently, a cutting plane algorithm was introduced [1] which applies to problems of the general form

$$\left.\begin{aligned} &\max_{x \epsilon X} f(x) \\ &\text{subject to } g(x, y) \geq 0 \text{ for all } y \epsilon Y \end{aligned}\right\} \text{Problem Q}$$

and hence applies to the problem which we are considering. The most interesting feature of the algorithm is that it generates a sequence of (potentially) nonconvex programs whose solutions will converge to a solution of Problem Q if the functions f and g are continuous over X and $X \otimes Y$, respectively, and if these sets are compact. While the results of [1] are contingent on the continuity of f and g, an examination of the proofs contained therein shows that the results are still valid with continuity replaced by upper semi-continuity of f and g in x and lower semi-continuity of g in y. Thus the algorithm will apply to problems of our form if we add lower and upper bounds l_0 and L_0 on the introduced variable x_0.

The cutting plane algorithm may be summarized as follows:

(1) Select $y^0 \epsilon Y$ and set $Y^0 = \{y^0\}$, $i = 0$.
(2) Solve the problem

$$\left.\begin{aligned} &\max_{x \epsilon X} f(x) \\ &\text{subject to } g(x, y^i) \geq 0 \text{ for each } y^i \epsilon Y^i. \end{aligned}\right\} \text{Problem } Q^i$$

Denote a solution by x^i.
(3) Solve the problem

$$\min_{y \in Y} g(x^t, y) \Big\} \text{Problem } R^t$$

Denote a solution by y^{t+1}.
(4) If $g(x^t, y^{t+1}) \geq 0$, stop. Otherwise set $Y^{t+1} = Y^t \cup \{y^{t+1}\}$ and continue.

Note that Problem Q^{t+1} has one more constraint that does Q^t. Under certain conditions which, unfortunately, do not apply in our case, all constraints $g(x, y^t) \geq 0$ $(j=1, \ldots, i)$ for which $g(x^t, y^j) > 0$ may be dropped from Problem Q^{t+1} without affecting convergence. We conjecture that such constraint dropping is possible in our case as well, but we have not tested this. Indeed, in the problems which we have run, convergence of the x^t's was obtained so quickly that none of the added constraints became nonbinding.

It is important to point out that the problems Q^t and R^t generated by the above scheme may be nonconvex problems, and hence may have proper local solutions. There are a few schemes now available which guarantee global solutions of problems of the form of Problem Q^t and R^t provided that the constituents of these problems are separable functions. In our case, Problem Q^t has the form

$$\max x_0$$

$$\text{subject to } x_0 \leq \sum_i \sum_j F_{ij}(x_i, y_i^k) \text{ for all } k \in Y^t$$

$$\sum_i f_i(x_i) \leq 0$$

$$l_0 \leq x_0 \leq L_0$$

$$l \leq x \leq L$$

and Problem R^t has the form

$$\min \sum_i \sum_j F_{ij}(x_i^t, y_j)$$

$$\text{subject to } \sum_j g_j(y_j) \leq 0$$

$$m \leq y \leq M,$$

so that all problems involved are separable. The particular scheme which we applied is described in detail in [3]. Briefly, a continuous separable problem of the form

$$\min \sum_{j=1}^{n} h_{0j}(x_j)$$

$$\text{subject to } \sum_{j=1}^{n} h_{ij}(x_j) \leq b_i \ (i=1, \ldots, m)$$

$$l_j \leq x_j \leq L_j$$

is replaced by its "piecewise linear approximating problem"

$$\min \sum_{j=1}^{n} \sum_{k=0}^{K_j} \theta_{jk} h_{0j}(x_{jk})$$

$$\text{subject to } \sum_{j=1}^{n} \sum_{k=0}^{K_j} \theta_{jk} h_{ij}(x_{jk}) \leq b_i \ (i=1, \ldots, m)$$

$$\sum_{k=0}^{K_j} \theta_{jk}=1 \ (j=1, \ldots, n)$$

$$\theta_{jk} \geq 0 \ (k=0, \ldots, K_j; \ j=1, \ldots, n)$$

where the x_{jk}'s are fixed grid points lying between l_j and L_j. The solution of this linear program offers a lower bound on the desired solution value. However, unless each set $\theta_j=\{\theta_{j0}, \ldots, \theta_{j\,K_j}\}$ consists of at most two nonzero and adjacent variables, the point

$$x_j=\sum_{k=0}^{K_j} \theta_{jh} x_{jh} \ (j=1, \ldots, n)$$

is not an approximate solution of the desired solution. A branch-and-bound scheme is then invoked which eventually insures that the sets θ_j do satisfy the adjacency conditions.

The modification necessary to handle lower semi-continuous functions $h_{ij}(x_j)$ is explained in [3].

The number of grid points that one selects is a user input. Increasing the number of grid points will, of course, increase the accuracy of the approximation, but will also increase the number of columns of the linear programs which are solved as subproblems. The total number of columns (exclusive of slack or artificial columns) of the subproblems is

$$\sum_{j=1}^{n} (K_j+1),$$

so that the column size of the L.P.'s is a linear function of the number of grid points chosen.

In practice, we have usually taken equally spaced grid points of about 50/original problem variable. To test the accuracy of our solutions, we then double or triple this number to test the sensitivity of the global value.

It should be emphasized that this global optimizing scheme will generate points which are only approximately equal to the desired solution, with the degree of approximation dependent on the number of grid points x_{jk} selected in $[l_j, L_j]$. An exact globally optimizing solution could usually be found by using any of the locally convergent algorithms (e.g., SUMT) by starting at the approximate solution.

4. SAMPLE PROBLEM

The problem considered here includes all of the extensions covered in Section 2. There is a search system y_1 capable of attacking percentage vulnerable systems x_1 and x_2 and a search system y_2 capable of attacking numerically vulnerable systems x_3 and x_4. In addition, there is a search system y_3 capable of attacking a bomber system x_5. The vulnerabilities a_1 and a_2 of missile systems x_1 and x_2 are allowed to be functions of the search effort y_1. The cost curves used here have jumps (discontinuities) due to initial procurement costs and also have shifts in their slopes reflecting large quantity cost discounts. An additional constraint on the y variable limits the total first attack force. There are constraints on the x variables that would be imposed by SALT agreements. The specific numbers used here were chosen to be fair y representative in size, but not in value, to actual numbers that might arise in a real application. The specific cost functions used do not reflect any actual cost functions, but were chosen to reflect the versatility of the model.

The sample problem is

$$\max_x \min_y \{5x_1e^{-a_1(y_1)\cdot y_1}+2x_2e^{-a_2(y_1)\cdot y_1}+2x_3e^{-0.5y_2/x_3}+x_4e^{-y_2/x_4}+0.5x_5(1-0.3 \arctan 0.02y_3)\}$$

where

$$a_1(y_1)=\begin{cases}0.02, & \text{if } 0\leq y_1\leq 4\\ 0.2, & \text{if } y_1>4,\end{cases}$$

$$a_2(y_1)=\begin{cases}0.04, & \text{if } 0\leq y_1\leq 4\\ 0.10, & \text{if } y_1>4.\end{cases}$$

Here x must satisfy

$$\sum_{i=1}^{5} f_i(x_i)\leq 2500$$

$$35x_1+30x_2\leq 360$$

$$10x_3+24x_4\leq 240$$

when

$$f_1(x_1)=\begin{cases}0, & \text{if } x_1=0\\ 200+60x_1, & \text{if } 0<x_1\leq 8\\ 320+45x_1, & \text{if } 8<x_1,\end{cases}$$

$$f_2(x_2)=\begin{cases}10x_2, & \text{if } 0\leq x_2\leq 5\\ 50x_2-200, & \text{if } 5<x_2,\end{cases}$$

$$f_3(x_3)=\begin{cases}8x_3, & \text{if } 0\leq x_3\leq 25\\ 38x_3-750, & \text{if } 25<x_3,\end{cases}$$

$$f_4(x_4)=\begin{cases}0, & \text{if } x_4=0\\ 37x_4+100, & \text{if } 0<x_4,\end{cases}$$

$$f_5(x_5)=\begin{cases}5x_5, & \text{if } 0\leq x_5\leq 10\\ 25x_5-200, & \text{if } 10<x_5.\end{cases}$$

The variables y_j must satisfy

$$\sum_{j=1}^{3} g_j(y_j)\leq 250$$

$$y_1+y_2+y_3\leq 55,$$

where

$$g_1(y_1)=17.5y_1,$$

$$g_2(y_2)=\begin{cases}5y_2/2, & \text{if } 0\leq y_2\leq 20\\ 25y_2/2-200, & \text{if } 20<y_2,\end{cases}$$

$$g_3(y_3)=\begin{cases}3y_3, & \text{if } 0\leq y_3\leq 10\\ 11y_3-80, & \text{if } 10<y_3\leq 15\\ 6y_3+10 & \text{if } 15<y_3.\end{cases}$$

These cost functions f_i and g_j indicate the various different possible slopes that can be assumed. For example, g_1 has a jump at zero reflecting a fixed or buy-in cost and a shift in slope at 8, reflecting per unit quantity discounts. The cost g_3 reflects only operating costs up to 10 units, after which procurement costs are added, again with quantity discounts allowed after 15.

The algorithm then sets up Problem Q^0:

$$\max_{x \epsilon X} x_0$$

$$\text{subject to } x_0 \leq F(x, y^0),$$

where the set X indicates those above constraints on x together with upper bounds 12, 15, 40, 40, 50 on $x_1, \ldots, x_5$ (their lower bounds are 0), and bounds $0 \leq x_0 \leq 200$. GLØBAL was then applied by requesting 20 subintervals on each of the variables $x_0, x_1, \ldots, x_5$. GLØBAL terminated after solving four linear programming problems with a global solution

$$(x_0^0, x_1^0, \ldots, x_5^0) = (103.86, 10.28, 0.0, 24.0, 0.0, 50.0).$$

The next step in the algorithm involves setting up the Problem R^0:

$$\min_{y \epsilon Y} F(x^0, y) - x_0^0$$

where Y includes all of the previous constraints on y together with upper bounds of 50 on y_1, y_2 and y_3. On this problem, GLØBAL terminated after a single LP solution with the global solution $y_1 = (10, 22, 0)$ and objective function value -68.49.

To form Problem Q^1, we now add the constraint

$$x_0 \leq F(x, y^1)$$

to those of Problem Q^0 and resolve. After solving four LP problems, GLØBAL terminated with the solution $(x_0^1, \ldots, x_5^1) = (65.040, 0.000, 13.167, 24.000, 0.000, 50.000)$.

Problem R^1 is then

$$\min_{y \epsilon Y} \{F(x^1, y) - x_0^1\}$$

whose solution was again found after solving a single LP problem to be $y^2 = (10, 22, 0)$. Since the optimal objective function value is 0 this time, the algorithm terminates.

The computational times for Problems Q^0, R^0, Q^1 and R^1 were 26.83, 14.99, 28.30 and 14.68 seconds for a total solution time of 84.80 seconds.

Several similar problems were run with different values of the coefficients. In none of these cases did the sequence Q^0, R^0, . . . proceed past R^1, even though the convergence theorem can, at best, guarantee limiting convergence.

It is difficult to make generalizations about the running times that one would expect on problems of this form, since such times are functions of:

(a) the number of original problem variables

(b) the number of grid points chosen

(c) the specific nature of the problem.

Item (c) is the most troublesome to address, since it dictates both the number of cuts that must be adjoined to the master problem and the number of nodes that are necessary to explore on the branch and bound tree.

A recent sequence of runs with NUGLØBAL on a single, badly behaved non-convex optimization problem with 14 original problem variables produced solutions requiring from 7.6 seconds to 41.6 seconds CPU on the university's IBM 370, model 145. A total of 15 problems (differing

only in the values of their input data, and not their functional forms) were solved in an average of 17.2 seconds of CPU time. Details of this study are found in reference [2].

REFERENCES

[1] Blankenship, Jerry W. and James E. Falk, "Infinitely Constrained Optimization Problems," Journal of Optimization Theory and Applications, *19*(2) (1976).

[2] Bracken, J., J. E. Falk and F. A. Miercort, "A Strategic Weapons Exchange Model," Technical Paper Serial T–325, Institute for Management Science and Engineering, The George Washington University, Washington, D.C. (November, 1975).

[3] Falk, James E., "An Algorithm for Locating Approximate Global Solutions of Nonconvex Separable Problems," Technical Paper Serial T–262, Program in Logistics, The George Washington University, Washington, D.C. (April, 1972).

[4] Shere, Kenneth and J. Wingate, "Allocation of Resources to Offensive Strategic Weapon Systems," Naval Research Logistics Quarterly, *23*(1) (1976).

SINGLE SERVER QUEUES WITH HYPEREXPONENTIAL SERVICE TIMES*

Irwin Greenberg
Briarcliff Manor, New York

ABSTRACT

Single server queues with hyperexponential service times are studied. Explicit expressions are derived for the waiting time and state distributions.

THE WAITING TIME

The queueing system under consideration has an interarrival time probability density function $a(t)$ with mean $1/\lambda$ and Laplace-Stieltjes transform $A^*(s)$. The service time is hyperexponential with density function

$$b(t)=2q^2\mu e^{-2q\mu t}+2(1-q)^2\mu e^{-2(1-q)\mu t}, \qquad 0<t$$

where $0<q\leq\frac{1}{2}$. The mean service time is $1/\mu$ and the Laplace-Stieltjes transform is

$$B^*(s)=B_1^*(s)/B_2^*(s) \tag{1}$$

where $B_1^*(s)=4q(1-q)\mu^2+2\mu[q^2+(1-q)^2]s$ and $B_2^*(s)=4q(1-q)\mu^2+2\mu s+s^2$. The traffic intensity is $\rho=\lambda/\mu$, which must be less than unity. Entry into service at the single server is in order of arrival.

Smith [5] treats single server queues with $\rho<1$, service in the order of arrival, and service time distributions whose Laplace-Stieltjes transforms can be written as the ratio of two polynomials, as in equation (1). The order of $B_1^*(s)$ is n and the order of $B_2^*(s)$ is at most $n-1$. The functional equation

$$B_2^*(s)-B_1^*(s)A^*(-s)=0 \tag{2}$$

has exactly n roots with negative real parts, call them $s_1, s_2, \ldots, s_n$. The Laplace-Stieltjes transform of the stationary waiting time distribution function, $F(t)$ is

$$\int_0^\infty e^{-st}dF(t)=\frac{B_2^*(s)}{B_2^*(0)}\prod_{j=1}^{n}\frac{s_j}{s_j-s}, \qquad Re(s)>0.$$

*This study was performed at the Graduate School of Public Administration of New York University under Contract number N00014-67-A-0028 sponsored by the Office of Naval Research.

For hyperexponential service times $n=2$ and the transform can be written (with $\beta=q(1-q)$)

$$\int_0^\infty e^{-st}dF(t)=\frac{s_1s_2}{4\beta\mu^2}\left\{1+\frac{(2\mu+s_1+s_2)s+(4\beta\mu^2-s_1s_2)}{(s-s_1)(s-s_2)}\right\},\ Re(s)>0. \tag{3}$$

Inverting this transform one finds the probability of zero wait to be

$$f(0)=\frac{s_1s_2}{4\beta\mu^2}. \tag{4}$$

The remainder of the inversion depends upon the nature of the roots s_1 and s_2.

Equation (2) can be specialized for the case of hyperexponential service:

$$4\beta\mu^2+2\mu s+s^2-A^*(-s)\{4\beta\mu^2+2\mu[q^2+(1-q)^2]s\}=0. \tag{5}$$

Since $q\leq\frac{1}{2}$, $\beta\leq\frac{1}{4}$. For $\beta<\frac{1}{4}$ $\left(\beta=\frac{1}{4}\right.$ corresponds to an exponential service time distribution$\left.\right)$ the left hand side of (5) is equal to $2\mu^2(1-2\beta+\sqrt{1-4\beta})\sqrt{1-4\beta}A^*(\mu+\mu\sqrt{1-4\beta})$ at $s=-\mu(1+\sqrt{1-4\beta})$ and is equal to $-2\mu^2(1-2\beta-\sqrt{1-4\beta})\sqrt{1-4\beta}A^*(\mu-\mu\sqrt{1-4\beta})$ at $s=-\mu(1-\sqrt{1-4\beta})$. Furthermore, $s=0$ is a root of equation (5) and the derivative at $s=0$ is $4\beta\mu(1-1/\rho)<0$. Thus, the left hand side of equation (5) is positive for $s=0-$, is negative for $s=-\mu(1-\sqrt{1-4\beta})$ and is positive for $s=-\mu(1+\sqrt{1-4\beta})$. Hence, if $A^*(s)$ is continuous for $-\mu(1+\sqrt{1-4\beta})<s<0$, the two roots with negative real parts that satisfy (5), s_1 and s_2, are real and distinct; one is between $-\mu(1+\sqrt{1-4\beta})$ and $-\mu(1-\sqrt{1-4\beta})$; the other is between $-\mu(1-\sqrt{1-4\beta})$ and zero. They can both be found by means of a simple root finding algorithm, for example, see Hildebrand [2].

For simplicity call these roots $s_1=-U$ and $s_1=-V$ so that U and V are positive real numbers with $V<U$. The inversion of the transform in (3) can now be completed:

$$f(t)=c_1e^{-Vt}+c_2e^{-Ut},\qquad t>0 \tag{6}$$

where $c_1=UV(4\beta\mu^2-V(2\mu-V))/4\beta\mu^2(U-V)$ and $c_2=-UV(4\beta\mu^2-U(2\mu-U))/4\beta\mu^2(U-V)$. Thus the waiting time in a single server queue with hyperexponential service is a mixture of exponentials. This generalizes the result for the waiting time in a single server queue with exponential service which is exponential.

Equation (6) is a special case of Theorem 4 of Smith [5]. In the more general case (i.e., service time distributions whose Laplace-Stieltjes transforms can be written as the ratio of two polynomials) the roots satisfying equation (2) need not be real as they are in the case of hyperexponential service times. The importance of queueing systems with hyperexponential service times has been commented upon by Morse [4] who suggested their use to approximate less tractable service time distributions whose standard deviations exceed their means.

It can be shown, after some tedious simplification, that

$$f(0)+\frac{c_1}{V-2q\mu}+\frac{c_2}{U-2q\mu}=f(0)+\frac{c_1}{V-2(1-q)\mu}+\frac{c_2}{U-2(1-q)\mu}=0. \tag{7}$$

This result will be needed in the next section.

THE NUMBER IN THE SYSTEM

Let $h(n; t)$ be the probability of exactly n arrivals to the system in time t commencing at the instant following an arrival. Jewell [3] has shown that

$$\int_0^\infty e^{-st}h(n; t)dt=\frac{1-A^*(s)}{s}\,[A^*(s)]^n,\ n=0, 1, 2, \ldots. \tag{8}$$

Haji and Newell [1] have shown that the probability of exactly n in the system at the instant following a departure in the steady state (and hence, at the instant preceding an arrival in the steady state) is

$$\pi_n=\int_0^\infty h(n; t)dG(t), \qquad n=0, 1, 2, \ldots \tag{9}$$

where $G(t)$ is the cumulative distribution function of the total time in the system, the waiting time plus the service time. It is the integral of the convolution of the service time density $b(t)$ with the waiting time "density" $f(t)$, including the mass at zero given by (4). That is, with the Dirac delta function denoted by $\delta(x)$,

$$\begin{aligned} dG(t)&=\left\{\int_0^t [f(0)\delta(x)+c_1e^{-Vx}+c_2e^{-Ux}]\cdot[2q^2\mu e^{-2q\mu(t-x)}+2(1-q)^2\mu e^{-2(1-q)\mu(t-x)}]dx\right\}dt \\ &=\left\{2\mu c_1\left[\frac{q^2}{2q\mu-V}+\frac{(1-q)^2}{2(1-q)\mu-V}\right]e^{-Vt}+2\mu c_2\left[\frac{q^2}{2q\mu-U}+\frac{(1-q)^2}{2(1-q)\mu-U}\right]e^{-Ut}\right\}dt. \end{aligned} \tag{10}$$

Equation (7) is utilized in the simplification leading to the final form of (10).

Substituting equation (10) into (9) and utilizing (8) results in

$$\begin{aligned} \pi_n=&\frac{2\mu c_1}{V}\left[\frac{q^2}{2q\mu-V}+\frac{(1-q)^2}{2(1-q)\mu-V}\right][1-A^*(V)][A^*(V)]^n \\ &+\frac{2\mu c_2}{U}\left[\frac{q^2}{2q\mu-U}+\frac{(1-q)^2}{2(1-q)\mu-U}\right][1-A^*(U)][A^*(U)]^n, \qquad n=0,1\ 2, \ldots. \end{aligned} \tag{11}$$

Here, $A^*(V)$ and $A^*(U)$ are the Laplace-Stieltjes transform of the interarrival distribution with the parameter s replaced by V and U respectively.

Let $h'(n; t)$ be the probability of exactly n arrivals to the system in time t. Jewell [3] has shown that

$$\begin{aligned} \int_0^\infty e^{-st}h'(n; t)dt&=\frac{1}{s}\left[1-\lambda\,\frac{1-A^*(s)}{s}\right], && n=0 \\ &=\lambda\left[\frac{1-A^*(s)}{s}\right]^2[A^*(s)]^{n-1}, && n=1, 2, \ldots. \end{aligned} \tag{12}$$

Haji and Newell [1] have shown that the probability of exactly n in the system at a random instant in the steady state is

$$p_n=\int_0^\infty h'(n; t)dG(t), \qquad n=0, 1, 2, \ldots.$$

Proceeding in the same manner as earlier,

$$p_0=1-\rho$$

$$p_n=\frac{2\mu c_1}{V^2}\left[\frac{q^2}{2q\mu-V}+\frac{(1-q)^2}{2(1-q)\mu-V}\right]\lambda[1-A^*(V)]^2[A^*(V)]^{n-1}$$

$$+\frac{2\mu c_2}{U^2}\left[\frac{q^2}{2q\mu-U}+\frac{(1-q)^2}{2(1-q)\mu-U}\right]\lambda[1-A^*(U)]^2[A^*(U)]^{n-1}, \qquad n=1, 2, \ldots.$$

Thus, the distributions of the number in the system are mixtures of geometric distributions when the service time distribution is hyperexponential. When the service time distribution is exponential, the distributions of the number in the system are simple geometric distributions.

EXPECTED VALUES

The expected waiting time of an arrival prior to the start of service is

$$W_Q=\int_0^\infty tf(t)dt=\int_0^\infty t(c_1e^{-Vt}+c_2e^{-Ut})dt=\frac{c_1}{V^2}+\frac{c_2}{U^2}$$

which follows from equation (6). The expected time in the system is $W=W_Q+1/\mu$, the expected number in the system is $L=\lambda W$, and the expected number waiting but not being served is $L_Q=\lambda W_Q$.

A SPECIAL CASE—HYPEREXPONENTIAL ARRIVALS

While in general it requires two applications of a root finding algorithm to determine U and V, only one application is necessary when the interarrival time is also a hyperexponential random variable with density function

$$a(t)=2p^2\lambda e^{-2p\lambda t}+2(1-p)^2\lambda e^{-2(1-p)\lambda t}, \qquad 0<t,$$

where $0<p\leq\frac{1}{2}$. The Laplace Stieltjes transform is

$$A^*(s)=[4\alpha\lambda^2+2\lambda(p^2+(1-p)^2)s]/(4\alpha\lambda^2+2\lambda s+s^2)$$

where $\alpha=p(1-p)$. Substituting this in (5) with s replaced by $-s$ and simplifying yields

$$s\{s^3+2(\mu-\lambda)s^2+4[\alpha\lambda^2-\beta\mu^2-2(\alpha+\beta)\lambda\mu+4\alpha\beta\lambda\mu]s-16\alpha\beta\lambda\mu(\mu-\lambda)\}=0. \tag{13}$$

There is one root in addition to s_1, s_2, and $s=0$. Since there is only one change of sign in equation (13), there is one real, positive, root—call it z. Thus, equation (13) can be written

$$s(s-z)(s+U)(s+V)=0.$$

Expanding the left hand side and equating coefficients of powers of s to the appropriate term in (13) yields

$$2(\mu-\lambda)=U+V-z$$

$$16\alpha\beta\lambda\mu(\mu-\lambda)=UVz.$$

Solving these for U and V yields

$$U, V=\frac{z}{2}+\mu-\lambda\pm\sqrt{\left(\frac{z}{2}+\mu-\lambda\right)^2-\frac{16\alpha\beta\lambda\mu(\mu-\lambda)}{z}}. \tag{14}$$

Thus, both U and V can be found if z is known. It can be seen that at $s=\lambda(1-\sqrt{1-4\alpha})$ the left hand side of (13) is negative while at $s=\lambda(1+\sqrt{1-4\alpha})$ it is positive. Thus, z falls between these values and can be found by a single application of the root finding algorithm. Then, (14) can be used to find U and V.

REFERENCES

[1] Haji, R. and G. F. Newell, "A Relation Between Stationary Queue and Waiting Time Distributions," Journal of Applied Probability *8*, 617–620 (1971).

[2] Hildebrand, F. B., *Introduction to Numerical Analysis* (McGraw-Hill, New York, 1956).

[3] Jewell, W. S., "The Properties of Recurrent-Event Processes," Operations Research, *8*, 446–472 (1960).

[4] Morse, P. M., *Queues, Inventories and Maintenance* (John Wiley and Sons, New York, 1958).

[5] Smith, W. L., "On the Distribution of Queueing Times," Cambridge Philosophical Society Proceedings, *49*, 449–461 (1953).

PROBABILISTIC PRIORITIES IN THE M/G/1 QUEUE

D. E. Matthews

University of Waterloo
Waterloo, Ontario, Canada

ABSTRACT

A simple method is presented for deriving the mean and variance of the queueing time distribution in an M/G/1 queue when the priorities assigned to customers have an assignment probability distribution. Several examples illustrate the results. The mean and variance of the queueing time distribution for the longest service time discipline are derived, and its disadvantages are discussed.

In one of the earliest considerations of the problem of queueing times in non-preemptive riority disciplines for the M/G/1 queue, Kesten and Runnenburg [5] derived expressions for the rst two moments of the queueing time, W_q, for customers in the rth class, C_r, of a k-class scheme, $1 \leq r \leq k$). These moments are specified by the formulae

1) $$E(W_q|C_r)=\frac{1}{2}\sum_{i=1}^{k}\lambda_i g_{i,2}/(1-\eta_{r-1})(1-\eta_r),$$

2) $$E(W_q^2|C_r)=\frac{1}{3}\sum_{i=1}^{k}\lambda_i g_{i,3}/(1-\eta_{r-1})^2(1-\eta_r)+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i g_{i,2}\right)\left(\sum_{i=1}^{r}\lambda_i g_{i,2}\right)\Big/(1-\eta_{r-1})^2(1-\eta_r)^2$$
$$+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i g_{i,2}\right)\left(\sum_{i=1}^{r-1}\lambda_i g_{i,2}\right)\Big/(1-\eta_{r-1})^3(1-\eta_r),$$

here λ_i is the Poisson arrival rate for customers in C_i, $g_{i,j}$ is the jth moment about the origin of ıe service time distribution for those same customers, and $\eta_j=\sum_{i=1}^{j}\lambda_i g_{i,1}$ $(1 \leq j \leq k)$. Cobham [1] had reviously obtained an expression identical to (1), and by considering the limiting case of Cobham's esult when each service time $t \epsilon [0, \infty)$ constitutes a distinct non-preemptive priority class, Phipps 3] evaluated the mean queueing time for the shortest service time queue discipline. But by relaxing 'hipps' assumption that priorities and service times coincide, additional interesting results may be btained, as the following argument shows.

Suppose that customers who join an M/G/1 queue which is in equilibrium are assigned non-reemptive priorities $S_1, S_2, \ldots$ which are independent, identically distributed observations from priority assignment probability distribution; the distribution is arbitrary up to monotonic trans-ɔrmation. For convenience we assume that $P(s)=pr(S_i \leqq s)$ $(i=1, 2, \ldots ; s \geqq 0)$, λ is the customer

arrival rate, and g_j ($j=1, 2, \ldots$) is the jth moment of the service time distribution, which is represented by the density function $g(x)$. Whenever a customer departs, the next customer to be served is always the one whose priority is greatest, i.e., the customer whose assigned priority is numerically smallest. However priorities are assigned, e.g., randomly or systematically, we shall require that $\mu_{j,s}=E(X_s^j)$ ($j=1, 2; s\geqq 0$) can be evaluated, where $\mu_{j,s}$ is the conditional jth moment about the origin of the service time, X_s, of customers belonging to the priority class with index $s \in [0, \infty)$. Let $\mu_s \equiv \mu_{1,s}$, and suppose that s is a continuity point of $P(.)$. Consider the limiting form of (1) and (2) in these circumstances. Clearly, the continuous analogues of λ_i and $g_{i,j}$ are $\lambda dP(s)$ and $\mu_{j,s}$, respectively, and it straightway follows that the first two moments of the queueing time W_q, for a customer with priority s are given by

$$E(W_q|s)=\frac{1}{2}\lambda g_2 \Big/ \Big\{1-\lambda\int_0^s \mu_t dP(t)\Big\}^2, \tag{3}$$

$$E(W_q^2|s)=\frac{1}{3}\lambda g_3 \Big/ \Big\{1-\lambda\int_0^s \mu_t dP(t)\Big\}^3 + \lambda^2 g_2 \Big\{\int_0^s \mu_{2,t} dP(t)\Big\} \Big/ \Big\{1-\lambda\int_0^s \mu_t dP(t)\Big\}^4. \tag{4}$$

By using Laplace transform techniques, both Takács [7] and Cohen ([2], p.454) derive expressions which agree with (3) and (4).

Integrate (3) and (4) with respect to s, the assigned priority. It follows that

$$E(W_q)=\frac{1}{2}\lambda g_2 \int_0^\infty dP(s) \Big/ \Big\{1-\lambda\int_0^s \mu_t dP(t)\Big\}^2, \tag{5}$$

(6)

$$E(W_q^2)=\frac{1}{3}\lambda g_3 \int_0^\infty \Big[1 \Big/ \Big\{1-\lambda\int_0^s \mu_t dP(t)\Big\}^3\Big] dP(s) + \lambda^2 g_2 \int_0^\infty \Big[\Big\{\int_0^s \mu_{2,t} dP(t)\Big\} \Big/ \Big\{1-\lambda\int_0^s \mu_t dP(t)\Big\}^4\Big] dP(s$$

We can use (5) and (6) to obtain both familiar and new results for the M/G/1 queue. The following three examples are important special cases.

THREE EXAMPLES

1. Shortest Service Time Discipline

Since (5) is a generalization of the result of Phipps [6] we obtain Phipps's expression for $E(W$ by setting $P(s)=G(s)$ and $E(X_s^j)=s^j$ ($j=1, 2; s\geqq 0$). Hence, if $g_j(t)=\int_0^t x^j g(x)\,dx$ ($j=1, 2; t>$ and $\lambda g_1(t)=\rho(t)<1$ for $t>0$,

$$E(W_q)=\frac{1}{2}\lambda g_2 \int_0^\infty [g(t)/\{1-\rho(t)\}^2]dt,$$

and

$$E(W_q^2)=\frac{1}{3}\lambda g_3 \int_0^\infty [g(s)/\{1-\rho(s)\}^3]ds + \lambda^2 g_2 \int_0^\infty [g_2(s)g(s)/\{1-\rho(s)\}^4]ds$$

may be used to evaluate $\mathrm{Var}(W_q)$.

. Random Priority Discipline

Suppose that the priorities assigned to customers are uniformly distributed over some fixed nterval, say [0, 1], and this assignment does not depend on service times. Obviously the equilibrium ueue size distribution is not affected by the priority assignment mechanism and hence the mean ueueing time will not differ from the value specified by the Pollaczek-Khintchine formula for ervice in order of arrival or service in random order. When the queue discipline is the latter, how-ver, at any service epoch each customer has an equal chance of being served next. Therefore, ueueing times should be more regular if customers are served in random order than if customers re served according to priorities assigned at random.

To ass'gn fixed priorities at random, independent of each cus omer's service time, let

$$P(s)=\begin{cases} s & 0\leqq s\leqq 1 \\ 1 & s>1 \end{cases}, \quad E(X_s^j)=g_j \qquad (j=1,\ 2;\ s\geqq 0).$$

fter simplifying, (5) and (6) become

$$E(W_q)=\frac{1}{2}\lambda g_2/(1-\rho),$$

7) $$\mathrm{Var}(W_q)=\frac{1}{6}\lambda g_3(2-\rho)/(1-\rho)^2+\frac{1}{12}\lambda^2 g_2^2(3+\rho)/(1-\rho)^3$$

here $\rho=\rho(\infty)=\lambda g_1<1$.

Note that the mean queueing time is identical to the Pollaczek-Khintchine formula, which ves the mean queueing time for service in random order. Cohen ([2], p. 431) indicates that the ariance of W_q for service in random order is

$$\frac{2}{3}\frac{\lambda g_3}{(1-\rho)(2-\rho)}+\frac{(\lambda g_2)^2(2+\rho)}{4(1-\rho)^2(2-\rho)}.$$

his is smaller than (7), indicating that queueing times are indeed more regular if customers are rved in random order than if customers are assigned fixed priorities at random.

Longest Service Time Discipline

Conway and Maxwell [3] mention a queue discipline which is the antithesis of the shortest rvice time rule—at each service epoch, always serve the customer with the longest service time. cording to these authors, this longest service time discipline can occur in practical situations. parently, a customer's importance is associated with the length of his service time; therefore, ger jobs are given priority over shorter ones.

Since a customer's priority is inversely proportional to his service time, $S\leqq s$ if and only if $\geqq s^{-1}$, where S is the customer's assigned priority and X is his service time. Therefore, $s)=1-G(s^{-1})$ and (3) and (4) become

$$E(W_q|s)=\frac{1}{2}\lambda^2 g_2/\{1-\rho+\rho(s^{-1})\}^2,$$

$$E(W_q^2|s)=\frac{1}{3}\lambda g_3/\{1-\rho+\rho(s^{-1})\}^3+\lambda^2 g_2\{g_2-g_2(s^{-1})\}/\{1-\rho+\rho(s^{-1})\}^4.$$

The averaged first and second moments of W_q are given by

$$E(W_q)=\frac{1}{2}\lambda g_2\int_0^\infty [g(z)/\{1-\rho+\rho(z)\}^2]dz,$$

$$E(W_q^2)=\frac{1}{3}\lambda g_3\int_0^\infty [g(z)/\{1-\rho+\rho(z)\}^3]dz+\lambda^2 g_2\int_0^\infty [g(z)\{g_2-g_2(z)\}/\{1-\rho+\rho(z)\}^4]dz.$$

A simple modification of the elegant argument of Cox and Smith ([4], pp. 84, 85) shows that the longest service time discipline results in a mean queueing time which exceeds that induced by any other discipline which does not permit preemptions. When long waiting times are subject to severe economic penalties it is probably sensible to give priority to the customer with the longest service time. Obviously, this particular queue discipline increases queueing times in contrast to, say, service in order of arrival. Although the longest service time rule can occur in practice, this queue discipline has been given little attention in the literature.

NUMERICAL RESULTS

To illustrate some of the differences among the queue disciplines considered in the previous section, expressions for $E(W_q)$ and $\mathrm{Var}(W_q)$ have been evaluated numerically for three different service time distributions and a selection of values of ρ. The distributions used were

$$D_1\colon\ g(x)=xe^{-x},$$

$$D_2\colon\ g(x)=\frac{1}{2}e^{-\frac{1}{2}x},$$

$$D_3\colon\ g(x)=\frac{1}{2}\left(e^{-x}+\frac{1}{3}e^{-x/3}\right);$$

D_1, D_2 and D_3 are simple examples of service time distributions with coefficients of variation, τ, less than, equal to and greater than unity, respectively. Each distribution has mean 2; hence changes in the traffic intensity are obtained by adjusting λ. The results of the calculations are given in Tables I and II.

Perhaps the most striking aspect of Table I is the contrast between the entries for service in order of arrival and those for the longest service time rule. Clearly, if mean queueing times ever approach the indicated levels when customers are served in this way (in contrast to levels achievable under service in order of arrival) the justification for adopting the longest service time discipline must be economic.

The standard deviations presented in Table II highlight several important features of the queue disciplines under consideration. A comparison of the columns for shortest service time and service in order of arrival reflect the hidden disadvantages of the former discipline. Conway and Maxwell [3] have indicated that although this 'optimal' queue discipline minimizes mean queueing times overall, the reductions are achieved at the expense of customers whose service times are long. The shortest service time rule usually means that their queueing times will be excessively long as well. This disadvantage is severely aggravated in heavy traffic conditions and is revealed in the larger queueing time variances for the shortest service time rule under these circumstances.

TABLE I. Mean queueing time for four different queue disciplines when service times are D_1 (Erlang), D_2 (exponential) or D_3 (mixed exponential)

Service time	Traffic intensity ρ	Shortest service time	Service in order of arrival	Random priority service	Longest service Time
	0. 05	0. 077	0. 079	0. 079	0. 082
	0. 15	0. 249	0. 265	0. 265	0. 299
	0. 25	0. 449	0. 500	0. 500	0. 620
	0. 35	0. 689	0. 808	0. 808	1. 12
D_1	0. 45	0. 985	1. 23	1. 23	1. 92
	0. 55	1. 37	1. 83	1. 83	3. 34
	0. 65	1. 90	2. 79	2. 79	6. 11
	0. 75	2. 74	4. 50	4. 50	12. 7
	0. 85	4. 40	8. 50	8. 50	34. 9
	0. 95	10. 8	28. 5	28. 5	261
	0. 05	0. 103	0. 105	0. 105	0. 108
	0. 15	0. 326	0. 353	0. 353	0. 383
	0. 25	0. 578	0. 667	0. 667	0. 771
	0. 35	0. 871	1. 08	1. 08	1. 34
D_2	0. 45	1. 22	1. 64	1. 64	2. 22
	0. 55	1. 66	2. 44	2. 44	3. 67
	0. 65	2. 24	3. 71	3. 71	6. 35
	0. 75	3. 11	6. 00	6. 00	12. 2
	0. 85	4. 73	11. 3	11. 3	30. 1
	0. 95	10. 5	38. 0	38. 0	179
	0. 05	0. 128	0. 132	0. 132	0. 135
	0. 15	0. 403	0. 441	0. 441	0. 483
	0. 25	0. 709	0. 833	0. 833	0. 980
	0. 35	1. 06	1. 35	1. 35	1. 72
D_3	0. 45	1. 47	2. 05	2. 05	2. 87
	0. 55	1. 96	3. 06	3. 06	4. 79
	0. 65	2. 60	4. 64	4. 64	8. 39
	0. 75	3. 52	7. 50	7. 50	16. 4
	0. 85	5. 14	14. 2	14. 2	41. 1
	0. 95	10. 6	47. 5	47. 5	250

The magnitude of the queueing time variances for the longest service time discipline underes the importance of an earlier suggestion that the justification for serving customers in this ay must be economic. Clearly, by serving customers with the longest jobs first, virtually every stomer is considerably inconvenienced.

The entries in Table II indicate that, for a given service time distribution, the longest service ne discipline maximizes not only the mean but also the variance of the queueing time distribun. If this result is also true within the wider class of all non-preemptive queue disciplines, then e longest service time rule may be regarded, on these grounds, as the worst possible non-preempre queue discipline.

CKNOWLEDGMENTS

These results constitute part of a thesis approved for the degree of Ph. D. in the University London. The financial assistance of the Commonwealth Scholarship Commission in the United ngdom is gratefully acknowledged.

TABLE II. Queueing time standard deviation for four different queue disciplines when service times are D_1 (Erlang), D_2 (exponential) or D_3 (mixed exponential)

Service time	Traffic intensity ρ	Shortest service time	Service in order of arrival	Random priority service	Longest service time
	0.05	0.457	0.466	0.472	0.498
	0.15	0.835	0.881	0.925	1.08
	0.25	1.16	1.26	1.38	1.80
	0.35	1.52	1.68	1.96	2.84
D_1	0.45	1.99	2.19	2.79	4.48
	0.55	2.70	2.87	4.09	7.35
	0.65	3.97	3.90	6.43	13.0
	0.75	6.66	5.68	11.5	26.8
	0.85	14.5	9.74	26.6	75.9
	0.95	74.2	29.8	149	637
	0.05	0.641	0.657	0.666	0.692
	0.15	1.15	1.24	1.30	1.46
	0.25	1.56	1.76	1.94	2.34
	0.35	1.99	2.34	2.73	3.57
D_2	0.45	2.51	3.04	3.85	5.45
	0.55	3.29	3.97	5.61	8.61
	0.65	4.64	5.35	8.75	14.7
	0.75	7.51	7.75	15.5	28.8
	0.85	15.9	13.2	35.7	76.7
	0.95	79.2	40.0	199	577
	0.05	0.845	0.868	0.880	0.912
	0.15	1.50	1.63	1.71	1.93
	0.25	2.01	2.32	2.54	3.11
	0.35	2.52	3.06	3.56	4.73
D_3	0.45	3.11	3.95	4.99	7.22
	0.55	3.94	5.43	7.22	11.4
	0.65	5.34	6.90	11.2	19.4
	0.75	8.32	9.91	19.7	33.1
	0.85	17.1	16.7	45.0	102
	0.95	83.2	50.2	249	765

REFERENCES

[1] Cobham, A., "Priority Assignment in Waiting Line Problems," Journal of the Operation Research Society of America, *2*, 70–76 (1954).

[2] Cohen, J. W., *The Single Server Queue* (North Holland, Amsterdam, 1969).

[3] Conway, R. W. and W. L. Maxwell, "Network Dispatching by the Shortest-Operation Discipline," Operations Research, *10*, 51–73 (1962).

[4] Cox, D. R. and W. L. Smith, *Queues* (Methuen, London, 1961).

[5] Kesten, H. and J. Th. Runnenburg, "Priority in Waiting Line Problems. I and II" Koninklijk Nederlandse Akademie van Wetenschappen Proceedings, Series A (Amsterdam) *60*, 312–33 (1957).

[6] Phipps, T. E., Jr., "Machine Repair as a Priority Waiting-Line Problem," Journal of the Operations Research Society of America, *4*, 76–85 (1956).

[7] Takács, L., "Priority Queues," Operations Research, *12*, 63–74 (1964).

CHARACTERIZATION OF THE NUCLEOLUS FOR A CLASS OF n-PERSON GAMES

Guillermo Owen
Rice University
Houston, Texas

ABSTRACT

An inductive procedure is given for finding the nucleolus of an n-person game in which all coalitions with less than n-1 players are totally defeated. It is shown that, for such a game, one of three things may occur: (a) all players receive the same amount; (b) each player receives his quota, plus a certain constant (which may be positive, nerative, or zero); (c) the weakest player receives one half his quota, and the other players divide the remaining profit according to the nucleolus of a similar $(n-1)$-person game. It is also shown that the nucleolus of such a game yields directly the nucleolus of each derived game. An example is worked out in detail.

In this paper, we consider n-person games ($n \geq 3$) whose characteristic function satisfies

$$v(S)=0 \qquad \text{if } |S| \leq n-2$$

$$0 \leq v(S) \leq v(N) \qquad \text{if } |S|=n-1$$

here $|S|$ is the cardinality of S. Such games have been studied in some detail. Maschler [5] gives e structure of their bargaining sets. Lucas in two separate papers [3, 4] has given a near-symmetric lution and studied such games in partition function form. Owen [6] gives all discriminatory solu- ons. We will characterize their nucleolus.

We shall define

$$h=v(N)$$

$$\Delta_i=h-v(N-\{i\}) \qquad i=1, 2, \ldots, n$$

d assume, without loss of generality, that

$$\Delta_1 \leq \Delta_2 \leq \ldots \leq \Delta_n.$$

e will use the notation

$$[h; \Delta_1, \Delta_2, \ldots, \Delta_n]$$

an alternative way of representing the game v.

By (2), the Δ_i must satisfy

(6) $$0 \leq \Delta_i \leq h.$$

We will however, insist only that $\Delta_i \geq 0$. If, additionally, all $\Delta_i \leq h$, we shall say v is a *proper* game if any $\Delta_i > h$, then v is *improper*. We will also consider the possibility that $n=2$; such games will also be considered improper.

Our development will be based on the notion of a *balanced collection*. Essentially, a collection

$$\mathcal{B} = \{S_1, S_2, \ldots, S_m\}$$

of subsets of N is *balanced* if there exists a vector $(\beta_1, \beta_2, \ldots, \beta_m)$ and a number λ such that

(7) $$\sum_{k \mid i \in S_k} \beta_k = \lambda \qquad \text{for all } i \in N$$

(8) $$\beta_k > 0 \qquad \text{for all } k=1, \ldots, m.$$

The vector β is the *balancing vector*, and its components the *balancing coefficients*, for the collection $\mathcal{B}$.

It is easy to see that the empty collection $\emptyset$, and the collections $\{\emptyset\}$ and $\{N\}$ are balanced. Moreover, the union of two balanced collections is also balanced.

We shall see below that, for $(n, n\text{-}1)$ games, it suffices to consider collections of 1-person and $(n\text{-}1)$-person sets. Then

THEOREM 1. Considering only 1-person and $(n\text{-}1)$-person coalitions, the only balanced collections (other than $\emptyset$) are:

(a) $$\{N-\{1\}, N-\{2\}, \ldots, N-\{n\}\}$$

(b) $$\{\{1\}, \{2\}, \ldots, \{n\}\}$$

(c) $$\{\{i\}, N-\{i\}\} \qquad \text{for any } i=1, \ldots, n$$

and unions of two or more of these.

PROOF: It is easy to see that any of these collections is balanced, with balancing vectors $(1, 1, \ldots, 1)$, $(1, 1, \ldots, 1)$, and $(1, 1)$ respectively. Similarly their unions are balanced.

Suppose, then, that $\mathcal{B}$ is balanced, but does not contain all the 1-person sets, nor all the $(n-1)$-person sets. Let β be represented by the vector

$$(\alpha_1, \ldots, \alpha_n; \gamma_1, \ldots, \gamma_n)$$

where α_i is the balancing coefficient for $\{i\}$, with $\alpha_i = 0$ if $\{i\} \in \mathcal{B}$; similarly, γ_i is the balancing coefficient for $N-\{i\}$, with $\gamma_i = 0$ if $N-\{i\} \in \mathcal{B}$. We know that at least one $\alpha_i = 0$, and at least one $\gamma_i = 0$.

Now, for each i, (7) takes the form

$$\sum_{k \mid i \in S_k} \beta_k = \alpha_i + \sum_{j \neq i} \gamma_j = \lambda$$

and so, adding γ_i to both sides,

$$\alpha_i + \sum_{j=1} \gamma_j = \lambda + \gamma_i$$

or

$$\gamma_i = \alpha_i + C$$

where $C = \Sigma\gamma_j - \lambda$ is independent of i.

Suppose, now, $C>0$. Since all $\alpha_i \geq 0$, this will mean $\gamma_i > 0$ for all i, contradicting our assumption that not all $(n-1)$-person sets are in $\mathcal{B}$. Similarly, if $C<0$, we woul dhave $\alpha_i = \gamma_i - C > 0$ for all i, contradicting the assumption that not all $\{i\} \in \mathcal{B}$. Thus $C=0$, and $\gamma_i = \alpha_i$. Thus, for each i, $\{i\} \in \mathcal{B}$ if and only if $N-\{i\} \in \mathcal{B}$, and so $\mathcal{B}$ is the union of collections of the type (c).

Define, now, an imputation for $[h; \Delta_1, \ldots, \Delta_n]$ as any vector $(x_1, \ldots, x_n)$ with non-negative components satisfying

$$x_1 + x_2 + \ldots + x_n = h.$$

For any imputation x, and coalition S, the excess of S in x is

$$e(S, x) = v(S) - \sum_{i \in S} x_i.$$

Given x, we define the 2^n-vector $\theta(x)$ by

$$\theta(x) = (e(S_1, x), e(S_2, x), \ldots, e(S_{2^n}, x))$$

where the S_k are the 2^n coalitions (subsets of N) taken in order of decreasing excess:

$$e(S_k, x) \geq e(S_{k+1}, x) \text{ for } k = 1, \ldots, 2^n - 1.$$

The nucleolus, $\tilde{x}$, is that imputation x for which $\theta(x)$ is *minimal in lexicographic ordering.* Schmeidler [7] proves the uniqueness of the nucleolus.

The following theorem is due to Kohlberg [1]. We give it without proof.

THEOREM 2. Let x be the nucleolus of v, and, for any q, let $\mathcal{B}$ be the set of coalitions S such that

$$e(S, x) \geq q.$$

Let $\mathcal{B}_0$ be the set of all $\{i\}$ such that $x_i = 0$. Then there exists a *balanced collection* $\mathcal{B}'$ such that

$$\mathcal{B} \subset \mathcal{B}' \subset \mathcal{B} \cup \mathcal{B}_0.$$

Conversely, if x is an imputation, and, for all q, the collection $\mathcal{B}$ has this property, then x is the nucleolus.

Essentially, Theorem 2 tells us that we can characterize the nucleolus in terms of balanced collections.

For our games, we have, if $|S| \leq n-2$,

$$e(S, x) = -\sum_{i \in S} x_i$$

and, in particular,

$$e(\{i\}, x) = -x_i.$$

On the other hand, for the $(n-1)$-person sets,

$$e(N-\{i\}, x) = v(N-\{i\}) - \sum_{j \neq i} x_j$$

$$= h - \Delta_i - (h - x_i)$$

$$e(N-\{i\}, x) = x_i - \Delta_i.$$

Since $x_i \geq 0$, we see that, for any S with $n-2$ players or less, and any $i \in S$,

$$e(S, x) \leq e(\{i\}, x)$$

and so, in Theorem 2 above, we see that, if $S \in \mathcal{B}$, we will also have $\{i\} \in \mathcal{B}$ for all $i \in S$. The following lemma will be of use:

LEMMA 1. Let $\mathcal{B}$ be a balanced collection, let $S \in \mathcal{B}$, and suppose $\{i\} \in \mathcal{B}$ for all $i \in S$. Then $\mathcal{B} - \{S\}$ is also balanced. Conversely, suppose $\{i\} \in \mathcal{B}$ for every $i \in T$. Then $\mathcal{B} \cup \{T\}$ is also balanced.

PROOF: To prove the first statement, let us suppose $S \in \mathcal{B}$ has the balancing coefficient β. We may remove S from $\mathcal{B}$ and increase the coefficient of each $\{i\} \subset S$ by the amount β; it is easy to check that (7)–(8) continue to hold.

Conversely, suppose that, for each $i \in T$, we have $\{i\} \in \mathcal{B}$ with the coefficient α_i. Choose $\beta > 0$, smaller than the minimum of these α_i; we can then add T to $\mathcal{B}$, with coefficient β, and change the coefficients α_i for each $i \in T$, to

$$\alpha_i' = \alpha_i - \beta > 0.$$

Again, it is easy to check that (4) and (5) hold.

In the light of the above comments, we can replace Kohlberg's criterion by a somewhat simpler one;

LEMMA 2. In Theorem 2, above, the conclusions still hold if the collection $\mathcal{B}$ is replaced by $\mathcal{B}''$ where $\mathcal{B}''$ consists of those 1-person and $(n-1)$-person sets which belong to $\mathcal{B}$.

PROOF: We know that, if $\mathcal{B}'$ contains any S, $2 \leq |S| \leq n-2$, then it must also contain all (i), $i \in S$. Then by Lemma 1, S can be removed from $\mathcal{B}'$ without affecting its balance. Thus we need not consider sets S with $2 \leq |S| \leq n-2$. Next, it can be seen that, if N or $\emptyset$ belong to $\mathcal{B}'$, these can also be removed from $\mathcal{B}'$ without affecting its balance (or lack of it). Hence $\mathcal{B}'$ will be balanced if and only if $\mathcal{B}''$ is balanced.

We are now in a position to give the first result for these games;

THEOREM 3. Let $\tilde{x}$ be the nucleolus for the game $[h; \Delta_1, \Delta_2, \ldots, \Delta_n]$. Then:

(a) If $\tilde{\Delta}_1 \geq 2h/n$, then

$$\tilde{x}_1 = \tilde{x}_2 = \ldots = \tilde{x}_n = \frac{h}{n}.$$

(b) If $\Delta_1 \geq 2\,\dfrac{\Sigma\Delta_j - h}{n}$, then

$$\tilde{x}_1 - \Delta_1 = \tilde{x}_2 - \Delta_2 = \ldots = \tilde{x}_n - \Delta_n = \frac{h - \Sigma\Delta_j}{n}.$$

(c) If neither of the two conditions, (a) or (b) holds, then

$$\tilde{x}_1 = \Delta_1/2.$$

PROOF: Suppose first (a) holds. Then

$$e(\{i\}, \tilde{x}) = -\frac{h}{n} \text{ for } i = 1, \ldots, n$$

and

$$e(N - \{i\}; \tilde{x}) = \tilde{x}_i - \Delta_i \leq \tilde{x}_i - \Delta_1 = \frac{h}{n} - \Delta_1 \leq = \frac{h}{n}.$$

Therefore all the 1-person sets have equal excess, $-\frac{h}{n}$, and none of the other coalitions has a larger excess. Thus the collections $\mathcal{B}''$ will be empty or contain the n 1-player sets, $\{1\}, \{2\}, \ldots, \{n\}$, and, by Theorem 1, will be balanced. Hence $\tilde{x}$ is the nucleolus.

Consider, next, case (b). In this case, we will have

$$e(N-\{i\}, \tilde{x})=\tilde{x}_i-\Delta_i=\frac{h-\Sigma\Delta_j}{n}$$

while, for any i,

$$e\{i\}, \tilde{x})=-\tilde{x}_i=-\Delta_i-\frac{h-\Sigma\Delta_j}{n}$$

$$\leq-\Delta_1-\frac{h-\Sigma\Delta_j}{n}\leq\frac{h-\Sigma\Delta_j}{n}.$$

Therefore, all the $(n-1)$-person sets have equal excess, $(h-\Sigma\Delta_j)/n$. All the 1-person sets, and *a fortiori* all other sets, have smaller excess. Thus the collections $\mathcal{B}''$ will either be empty or contain the n $(n-1)$-player coalitions, if they contain any other S, will also contain all $\{i\}\subset S$. Thus, by theorem 1, all $\mathcal{B}''$ are balanced. Hence $\tilde{x}$ is the nucleolus.

Finally, suppose neither of the two cases (a) or (b) holds. In this case, we can see that, for any y,

$$e(\{1\}, y)+e(N-\{1\}, y)=-y_1+y_1-\Delta_1$$

and thus it is not possible to have either of these two excesses smaller than $-\Delta_1/2$, without making the other one greater than $-\Delta_1/2$.

We note, next, that, since (a) does not hold, we have

$$\frac{n\Delta_1}{2}<h$$

and, since (b) does not hold,

$$\Sigma_{\Delta_i}-\frac{n\Delta_1}{2}>h.$$

Consider, next, the set of constraints

$$\frac{\Delta_1}{2}\leq y_j\leq\Delta_j-\frac{\Delta_1}{2};\ j=1, \ldots, n.$$

These constraints are feasible as $\Delta_j\geq\Delta_1$ for all j, and, since each y_j is allowed to range over its interval independently of the others, the sum y_j can take on any value in the interval.

$$\left[n\frac{\Delta_1}{1}, \Sigma_{\Delta_i}-\frac{n\Delta_1}{2}\right]$$

Since this interval contains h, we can choose y satisfying the constraints, such that $y_j=h$. Now, for this y,

$$e(\{i\}, y)=-y_i\leq-\frac{\Delta_1}{2}$$

and

$$e(N-\{i\}, y)=y_i-\Delta_i\leq-\frac{\Delta_1}{2}$$

and this, for all coalitions S, other than $S=N$ and $S=\phi$, $e(S, y) \geq -\Delta_1/2$.

Thus it is possible to find an imputation which makes all excesses smaller than or equal to $-\Delta_1/2$. But, as we saw above, this can only be done if both $\{1\}$ and $N-\{1\}$ have exactly this excess. Thus the nucleolus must satisfy $\tilde{x}_1=\Delta_1/2$.

In case (c), above, the nucleolus is not completely characterized; only the first component is determined. To see how we must continue, let us suppose that, for the game

$$v=[h; \Delta_1, \Delta_2, \ldots, \Delta_n]$$

one of the components x_i of an imputation x has been determined. Thus will leave an amount

$$h'=h-x_i$$

to be divided among the members of the set $M=N-\{i\}$, and thus the remaining components of x:

$$\hat{x}=(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

will be an imputation for the m-person game

$$u=[h'; \Delta_1, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n].$$

Let us use f to denote excesses in the game u. We will have, here,

$$f(M-\{j\}, \hat{x})=x_j-\Delta_j$$

$$f(\{j\}, \hat{x})=-x_j$$

for all $j \neq i$. Thus, for all $j \neq i$, we have

$$f(M-\{j\}, \hat{x})=e(N-\{j\}, x) \tag{9}$$

$$f(\{j\}, \hat{x})=e(\{j\}, x). \tag{10}$$

Therefore, the excesses for the $(m-1)$-person coalitions in u are equal to those for the correspondin $(n-1)$-person coalitions in v, and the excesses for the 1-person sets are the same in u as in v.

A certain problem arises, here, when $n=3$. In this case, the derived game u has only 2 player and hence is not strictly speaking, a game of the type we consider here. The real difficulty stem from the fact that, for $m=2$ the 1-person coalitions must simultaneously play the role of 1-perso and $(m-1)$-person sets. Now, for $n \geq 3$, we have seen that there are only $2n$ important coalitions- those with 1 or $n-1$ players; but for $n=2$, these reduce, not to $2n=4$, but only to 2 coalition We shall, however, avoid this problem by assuming:

AGREEMENT

For a 2-person game

$$u=[h; \Delta_1, \Delta_2]$$

the two 1-person coalitions will each be counted twice with the coalition $\{1\}$ appearing once the form $\{1\}$, with $u(\{1\})=0$, and once in the form $N-\{2\}$, with $u(N-\{2\})=h-\Delta_2$. Similarl $\{2\}$ will appear once in the form $\{2\}$, with $u(\{2\}=0$, and once in the form $N-\{1\}$, wi $u(N-\{1\})=h-\Delta_1$.

It should be noticed that, with this agreement, the equations (9) and (10) will continue to hold, even when $n=3$.

THEOREM 4. Let $x=(x_1, \ldots, x_n)$ be the nucleolus for the $(n, n-1)$ game $v=[h; \Delta_1, \ldots, \Delta_n]$. For any $i=1, \ldots, n$, let u be the $(n-1, n-2)$ game

$$u=[h'; \Delta_1, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n]$$

where $h'=h-x_i$. Then

$$\hat{x}=(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

is the nucleolus of u.

PROOF: For any q, let $\mathcal{B}$ be the set of coalitions $S\subset N$ with

$$e(S, x)\geq q,$$

let $\mathcal{B}''$ be the set of coalitions $S\in\mathcal{B}$ with $|S|=1$ or $n-1$, and let $\mathcal{B}_0$ be the set of $\{j\}$ such that $j-N$, $x_j=0$. We know there is a U',

$$\mathcal{B}''\subset\mathcal{B}'\subset\mathcal{B}''\cup\mathcal{B}_0$$

which is balanced.

Let, now, $\mathcal{A}$ be the set of coalitions $T\subset M$ with

$$f(T, \hat{x})\geq q,$$

let $\mathcal{A}''$ be the set of coalitions $T\subset\mathcal{A}$ with $|T|=1$ or $n-2$. By (9) and (10), it is easy to see that, for any $j\neq i$, $M-\{j\}\epsilon\,\mathcal{A}''$ if and only if $N-\{j\}\epsilon\,\mathcal{B}''$; $\{j\}\epsilon\,\mathcal{A}''$ if and only if $\{j\}\epsilon\mathcal{B}''$. Let α_0 consist of all $\{j\}$, $j\subset M$, such that $x_j=0$; clearly $\mathcal{A}_0=\mathcal{B}_0-\{j\}$.

Define, now, $\mathcal{A}'$ as $\mathcal{A}'=\mathcal{A}_1'\cup\mathcal{A}_2'$, where

$$\mathcal{A}_1'=\{M-\{j\}|j\neq i,\ N-\{j\}\epsilon\mathcal{B}'\}$$

$$\mathcal{A}_2'=\{\{j\}|j\neq i,\ \{j\}\epsilon\mathcal{B}'\}.$$

Since $\mathcal{B}''\subset\mathcal{B}'\subset\mathcal{B}''\cup\mathcal{B}_0$, we see that

$$\mathcal{A}''\subset\mathcal{A}'\subset\mathcal{A}''\cup\mathcal{A}_0.$$

We will now prove that $\mathcal{A}'$ is balanced. Since $\mathcal{B}'$ is balanced in N, and contains only 1-person and $(n-1)$-person sets, it must either be empty, or have one of the forms

(α) All $(n-1)$-person sets $N-\{j\}$, plus possibly some 1-person sets.

(β) All 1-person sets (j), plus possibly some $(n-1)$-person sets.

(γ) Exactly k 1-person sets $\{j_1\}, \ldots, \{j_k\}$, with their complements $N-\{j_1\}, \ldots, N-\{j_k\}$

Clearly, if $\mathcal{B}'$ is empty, so is $\mathcal{A}'$, and so $\mathcal{A}'$ is balanced. In the other cases

(α) $\mathcal{A}'$ contains all $n-1$ of the $(n-2)$-person sets $M-\{j\}$, and thus is balanced in M.

(β) $\mathcal{A}'$ contains all $n-1$ of the 1-person sets $\{j\}$, and thus is balanced in M.

(γ) $\mathcal{A}'$ contains either k or $k-1$ of the 1-person sets, together with their $(n-2)$-person complements in M, and thus $\mathcal{A}'$ is balanced in M.

In all cases, then, $\mathcal{A}'$ is balanced in M. Thus $\hat{x}$ is indeed the nucleolus for u.

Theorem 4 shows us, then, that the nucleolus, for these games, is hereditary: any k of the components of the nucleolus give us the nucleolus of a derived $(k, k-1)$ game. Our desire, however, is to do the opposite: build "up" from the nucleoli of the smaller derived games to that of the larger game. The following corollary shows how this is done

COROLLARY: With the notation of Theorem 3 above, suppose neither of the cases (a) or(b) holds. Then v has the nucleolus

$$\tilde{x}=\left(\frac{\Delta_1}{2}, \tilde{x}_2, \ldots, \tilde{x}_n\right)$$

where $(\tilde{x}_2, \ldots, \tilde{x}_n)$ is the nucleolus of the derived game

$$u=[h'; \Delta_2, \ldots, \Delta_n]; h'=h-\frac{\Delta_1}{2}.$$

The proof of this is immediate from Theorem 3 which tells us that we must have $\tilde{x}_1=\Delta_1/2$, and Theorem 4. This gives us, then, an inductive procedure for computing the nucleolus of the game v; we simply check whether case (a) or (b) will hold. In either case (a) or (b), the nucleolus is given immediately. If neither (a) nor (b) holds, we assign the value $\tilde{x}_1=\Delta_1/2$, and proceed to the derived game u. Thus, after at most n steps, the nucleolus will be obtained. An example will show how this is done.

EXAMPLE: Let us consider the 4-person game v given by:

$$v(\{1, 2, 3\})=0.3, v(\{1, 2, 4\})=0.5, v(\{1, 3, 4\})=0.8$$

$$v(\{2, 3, 4\})=0.9, v(\{1, 2, 3, 4\})=1$$

and $v(S)=0$ for all other S. This is represented by

$$v=[1; 0.1, 0.2, 0.5, 0.7].$$

We have

$$\frac{h}{n}=\frac{1}{4}=0.25; \frac{\Sigma\Delta_j-h}{n}=\frac{0.5}{4}=0.125; \frac{\Delta_1}{2}=0.05.$$

Thus neither (a) nor (b) holds, and we will have $\tilde{x}_1=0.05$.

We consider next the derived game

$$v'=[0.95; 0.2, 0.5, 0.7].$$

For this game, we have

$$\frac{h}{n}=\frac{0.95}{3}=0.317; \frac{\Sigma\Delta_j-h}{n}=\frac{0.45}{3}=0.15; \frac{\Delta_2}{2}=0.1.$$

Again, neither (a) nor (b) holds; we have therefore $\tilde{x}_2=0.1$.

We proceed, then, to the game

$$v''=[0.85; 0.5, 0.7].$$

For this game, we have

$$\frac{h}{n}=\frac{0.85}{2}=0.425; \frac{\Sigma\Delta_j-h}{n}=\frac{.35}{2}=.175; \frac{\Delta_3}{n}=0.25$$

and we see that case (b) holds. Thus we have

$$\tilde{x}_3 - 0.5 = \tilde{x}_4 - 0.7 = -.175$$

or equivalently, $\tilde{x}_3 = 0.325$, $\tilde{x}_4 = 0.525$. Thus

$$\tilde{x} = (0.05,\ 0.1,\ 0.325,\ 0.525)$$

is the nucleolus for the original game.

We note that, in doing this, we have obtained the nucleolus also for all of the derived games. Thus [0.9; 0.1, 0.5, 0.8] has the nucleolus (0.05, 0.325, 0.525); similarly [0.475; 0.1, 0.2, 0.5] (which is improper) has the nucleolus (0.05, 0.1, 0.325), and the improper game [0.575; 0.1, 0.7] has the nucleolus (0.05, 0.525).

NOTES

The following points are of interest:

Note 1

We have specifically insisted, throughout, that all $\Delta_i \geq 0$. If a game has any $\Delta_i < 0$, then the game is of course, improper (it contradicts condition (2)); since we have, however, allowed the possibility $\Delta_i > h$, which also contradicts condition (2), it seems overly strict to insist on $\Delta_i \geq 0$.

Suppose, then, that $\Delta_i < 0$. If, in Theorem 3, case (c) holds, we are told to set $x_1 = \Delta_1/2$, which is negative. This would mean that x is not an imputation, as, for these games, imputations are non negative. There are then two alternatives, either of which can be chosen:

(α) We can insist that the nucleolus must be an imputation. In this case, when (c) holds, we will simply set $x_1 = 0$, rather than $x_1 = \Delta_1/2$, and proceed as before. All of our results continue to be true.

(β) We can drop the non-negativity condition, and insist only that the components of the nucleolus must have sum h. In this case, unfortunately, our Lemma 2 is no longer valid, and so our procedure will not work. For example, in the game [1; −3, −3, 0, 0, 0] our procedure would give the vector (−1.5, −1.5, 1.33, 1.33, 1.33), whereas the nucleolus is (−1, −1, 1, 1, 1).

Note 2

It may be seen that the procedure given here for computing the nucleolus is simply Kopelowitz's procedure [2], which solves a sequence of linear programs. Due to the special form of our games, these l.p.'s are especially easy to solve here, thus yielding the result very quickly.

REFERENCES

[1] Kohlberg, E., "On the Nucleolus of a Characteristic Function Game," SIAM Journal of Applied Mathematics, *20*, 62–66 (1971).

[2] Kopelowitz, A., "Computation of the Kernels of Simple Games and the Nucleolus of N-Person Games," RM 31, Research Program in Game Theory and Mathematical Economics, Department of Mathematics, Hebrew University of Jerusalem (September 1967).

[3] Lucas, W. F., "n-Person Games with only 1, n-1, and n-Person Coalitions," Z. Zeitschrift der Wahrscheinlichkeitstheorie und Verwandte Gebiete, *6*, 287–292 (1966).

[4] Lucas, W. F., "Solutions for a Class of n-person Games in Partition Function Form," Naval Research Logistics Quarterly, *14*, 15–21 (1967).

[5] Maschler, M., "n-Person Games with only 1, n-1, and n-Person Permissible Coalitions," J. Math. Anal. Appl., *6*, 230–256, (1963).

[6] Owen, G., "n-Person Games with only 1, n-1, and n-Person Coalitions," Proceedings of the American Mathematical Society, *19*, 1258–1261 (1968).

[7] Schmeidler, D., "The Nucleolus of a Characteristic Function Game," SIAM Journal of Applied Mathematics, *17*, 1163–1170 (1969).

A PARALLEL SEQUENCING ALGORITHM FOR MINIMIZING TOTAL COST

M. K. Rajaraman
Texas Southern University
Houston, Texas

ABSTRACT

The problem of sequencing jobs on parallel processors when jobs have different available times, due dates, penalty costs and waiting costs is considered. The processors are identical and are available when the earliest job becomes available and continuously thereafter. There is a processor cost during the period when the processor is available for processing jobs. The proposed algorithm finds the sequence (or sequences) with minimum total cost (sum of waiting, penalty and processor costs). A proof of the algorithm and numerical results are given.

INTRODUCTION

Consider the situation where there are m identical processors and there are n jobs to be processed. Each job needs to be processed only once on any one of the m processors. The processors can process only one job at a time. The jobs have predetermined available times, processing times and due dates. There is a penalty cost for each job that finishes after due date and a job-waiting cost if the job waits for a processor to become available. There is also a processor cost that is incurred until the last job completes processing. The problem of parallel sequencing can be stated as follows:

Given m identical processors and n jobs which have predetermined processing times, available times and due dates, find a sequence that minimizes the sum of the job-lateness penalty cost, job-waiting cost and processor cost, when the jobs can be processed on any one of the processors and a processor can process only one job at a time.

There have been a number of articles in recent literature that deal with this problem. Rothkopf [6] has given some results on the solution of certain catagories of parallel sequencing problems. The main criterion used in his formulations is the discounted, linear waiting cost. Gupta and Maykut [1] have given a dynamic programming formulation for certain specific types of cost functions. The present author has given an algorithm for minimizing non-linear waiting costs [5]. A number of articles [2, 3, 4] have appeared on the single facility sequencing that will provide some insight into the parallel sequencing problem. However, there is no algorithm in the existing literature that deals with minimizing "total cost."

The criterion of "total cost" used here appears to be more realistic in the sequencing environment. Most of the existing algorithms consider minimizing total penalty cost, or mean flow time,

or average job tardiness, or maximum job lateness. While one can find situations where these criteria are effective, in general, they represent only one type of cost. Further, many of the existing algorithms assume that all jobs are simultaneously available at the beginning of the scheduling period. This has the effect of making the processor-use cost independent of the job sequence. When all the jobs are not available simultaneously, the completion time for processing all the jobs becomes sequence-dependent and the processor cost becomes relevant. Also, since the in-process inventory cost could amount to a large percentage of the value of the inventory, the waiting cost for the jobs needs to be considered. In view of these facts, the "total cost," as defined here, consists of the sum of penalty cost, job-waiting cost and processor cost.

FORMULATION OF THE PROBLEM

In formulating the parallel sequencing problem the following assumptions will be made:

(1) There are m identical processors and n different jobs, such that $n>m$.

(2) The jobs have predetermined available times, processing times and due dates.

(3) The m processors become available at the time the earliest job becomes available and continuously thereafter.

(4) Whichever processor becomes available earliest begins to process the next job in sequence.

(5) A job, once begun by a processor, is processed to completion (that is, job splitting is not allowed).

(6) Every job that is completed after its due date incurs a penalty cost which is monotone, non-decreasing.

(7) Every job that waits for a processor incurs a waiting cost which is monotone, non-decreasing.

(8) The m processors incur a use cost at a fixed rate (same for all processors) from the time the earliest job becomes available until all the jobs are processed.

(9) The job set-up times are assumed to be independent of the job sequence and are included in the processing time.

The following notations will be used in describing and proving the algorithm.

n=number of jobs

m=number of processors

x=processor use cost per unit time per processor

S_i=a partial sequence containing i jobs

S_i'=a partial sequence containing the same i jobs as S_i, but in a different order

$k_i(k_i')$=job number that is in the i^{th} position in $S_i(S_i')$

$a_{S_i}(a_{S_i'})$=earliest processor available time for sequence $S_i(S_i')$

$1_{S_i}(1_{S_i'})$=latest processor available time for sequence $S_i(S_i')$

$p(j)$=processing time for job j (j=1, 2, . . . n)

$A(j)$=available time for job j

$D(j)$=due date for job j

$r(j)$=penalty rate for job j

$q(j)$=waiting cost per unit time for job j

$T(j)$ = completion time for job j in S_t
$T'(j)$ = completion time for job j in S'_t
$W(j)$ = waiting time for job j
$d(j)$ = delay in the completion of job j
$h(S_t)$ = sum of waiting cost and penalty cost for all jobs in sequence S_t
$f(S_t)$ = processor cost for processing all jobs in sequence S_t
$g(S_t)$ = total cost for processing all jobs in sequence S_t

As there are m identical processors, the order of the first m jobs in any given sequence is immaterial. However, all possible combinations of m jobs out of n need be considered. The jobs in sequence positions beyond m will be assigned to the processors when both jobs and processors become available. Otherwise, either the processor or the job will have to wait. If more than one processor is available when a job arrives, it will be scheduled on the processor that has the longest idle time. With these assumptions, the following relationships are immediately obtainable.

(1) $$T(k_t)=A(k_t)+p(k_t) \text{ for } i\leq m$$

(2) $$T(k_t)=\text{Max } [a_{S_{i-1}}, A(k_t)]+p(k_t) \text{ for } i>m$$

(3) $$W(k_t)=0 \text{ for } i\leq m$$

(4) $$W(k_t)=\text{Max } [a_{S_{i-1}}-A(k_t), 0] \text{ for } i>m$$

(5) $$d(k_t)=\text{Max } [T(k_t)-D(k_t), 0]$$

(6) $$f(S_t)=m\cdot x\cdot 1_{S_t}$$

(7) $$a_{S_t}\leq 1_{S_t}$$

$$h(S_t)=\sum_{j=1}^{i}[W(k_j)q(k_j)+d(k_j)r(k_j)]$$

(8) $$=h(S_{t-1})+W(k_t)q(k_t)+d(k_t)r(k_t)$$

(9) $$g(S_t)=h(S_t)+f(S_t).$$

Also note that, for $i<m$ $\quad a_{S_i}=0; \quad 1_{S_i}=\underset{j}{\text{Max}}\ \{p(j)\}, \quad j\epsilon S_t$

for $i=m$,

$$a_{S_i}=\underset{j}{\text{Min}}\ \{A(j)+p(j)\},\ j\epsilon S_t$$
$$1_{S_i}=\underset{j}{\text{Max}}\ \{A(j)+p(j)\},\ j\epsilon S_t$$

for $i>m$ let $X_t=\text{Max } (a_{S_{i-1}}, A(i))+p(i)$ then, $a_{S_i}=\text{Min } \{X_t, 1_{S_{i-1}}\}$ and $1_{S_i}=\text{Max } \{X_t, 1_{S_{i-1}}\}$
The objective is to find an S_n that will minimize $g(S_n)$.

THE ALGORITHM

The algorithm begins with a set of $\binom{n}{m}$ partial sequences of length m, such that each sequence has a unique set of m jobs. From these sequences it generates new sequences of length $(m+1)$ by

taking each sequence of length m and adding, in turn, all the $(n-m)$ possible jobs to it. Thus there will be $\binom{n}{m}(n-m)$ sequences of length $(m+1)$. From these sequences we form *groups* such that each group has the same $(m+1)$ jobs, but in different order. In each group, we consider sequences in pair and eliminate one of the pair if a criterion (to be specified) is met by both sequences. Otherwise both sequences will be retained. This is repeated for all possible pairs in a group and for all groups. Note that paired comparison is done only within groups. With the remaining sequences of length $(m+1)$, we generate new sequences of length $(m+2)$. Thereafter, we form the groups once again, and go through the elimination procedure. Finally when we arrive at sequences of length n, there will be only one group which can be searched to find the minimum cost sequence. The steps of the algorithm are given below.

Step 1: Form sequences containing m of the n jobs, making sure that only one ordering of each of the m jobs is used in generating the sequences.

Step 2: Take a sequence from the last step and generate new sequences from it, by adding, in turn, each of the jobs not already in the sqeuence. Repeat this for all the sequences from the last step.

Step 3: Arrange the sequences from the last step into groups so that each group has all the sequences with the same set of jobs but in different permutations. Let the length of the sequence at this stage be i.

Step 4: For each of the sequences so arranged, calculate a_{S_i} and $g(S_i)$.

Step 5: Let S_i and S_i' denote two sequences from the *same group*. They will have the same set of jobs, but in different order.

Eliminate S_i' if $a_{S_i} \leq a_{S_i'}$ and $g(S_i) \leq g(S_i')$

Eliminate S_i if a $a_{S_i} > a_{S_i'}$ and $g(S_i) \geq g(S_i')$

If none of the above conditions is satisfied retain both S_i and S_i'.

Repeat this step for all pairs within all groups.

Step 6: If $i<n$ go to Step 2. If $i=n$, there will be only one group of sequences. From this group choose the sequence(s) that has the minimum $g(S_n)$.

PROOF OF THE ALGORITHM

We will present an inductive proof of the algorithm. Consider, for some i, two sequences S_i and S_i', both of which have the same set of jobs, but in different permutations. To obtain a sequence of length n from these, we could concatenate one possible permulation of the remaining $(n-1)$ jobs to both S_i and S_i'. Let S_{i+1} and S_{i+1}' be obtained from S_i and S_i' respectively by adding some job j to the sequences S_i and S_i'.

Let the conditions

$$a_{S_i} \leq a_{S_i'} \text{ and} \tag{10}$$

$$g(S_i) \leq g(S_i') \tag{11}$$

hold for S_i and S_i'. We will show that $g(S_{i+1}) \leq g(S_{i+1}')$ and therefore, by induction, $g(S_n) \leq g(S_n')$. Thus S_i' can be eliminated from the set of sequences. From equations (1) through (9) the following equations can be obtained.

$$T(k_{i+1})=\text{Max}\,[a_{S_i}-A(k_{i+1})]+p(k_{i+1}) \tag{12}$$

$$T'(k_{i+1})=\text{Max}\,[a_{S'_i}-A(k_{i+1})]+p(k_{i+1}) \tag{13}$$

Note that k_{i+1} is the same for both S_i and S'_i.

$$g(S_{i+1})=h(S_{i+1})+f(S_{i+1}) \tag{14}$$

$$g(S'_{i+1})=h(S'_{i+1})+f(S'_{i+1}) \tag{15}$$

$$h(S_{i+1})=h(S_i)+q(k_{i+1})\cdot\text{Max}\,[a_{S_i}-A(k_{i+1}),\,0]+r(k_{i+1})\cdot\text{Max}\,[T(k_{i+1})-D(k_{i+1}),\,0] \tag{16}$$

$$f(S_{i+1})=m\cdot x\cdot 1_{S_{i+1}} \tag{17}$$

Similar relations can be obtained for $h(S'_{i+1})$ and $f(S'_{i+1})$ by using S'_i, S'_{i+1} and T' in place of S_i, S_{i+1} and T in the above equation for h and f.

Since,

$$g(S_i)-g(S'_i)\leq 0,$$

$$h(S_i)-h(S'_i)\leq 0 \tag{18}$$

because h and f are monotone non-decreasing functions.

Therefore from (16)

$$h(S_{i+1})-h(S'_{i+1})\leq 0 \tag{19}$$

Also, since

$$1_{S_{i+1}}\geq 1_{S_i}\geq a_{S_i} \tag{20}$$

$$1_{S'_{i+1}}\geq 1_{S'_i}\geq a_{S'_i} \tag{21}$$

and $a_{S_i}\leq a_{S'_i}$ from (17), we obtain

$$f(S_{i+1})-f(S'_{i+1})\leq 0 \tag{22}$$

Therefore,

$$g(S_{i+1})\leq g(S'_{i+1}) \tag{23}$$

By induction on i, it follows that

$$g(S_n)\leq g(S'_n) \tag{24}$$

Thus S'_i will give rise to no better solution than S_i, and S'_i can be eliminated. If the inequalities (10) and (11) are reversed, it follows that S_i can be eliminated.

It should be noted that the algorithm essentially eliminates a certain number of partial sequences at each stage. The larger the number of partial sequences eliminated and the earlier in the process we eliminate them, the better will be the performance of the algorithm. If, for example, at any stage where the sequences are of length i, we eliminate one partial sequence, that discards $(n-i)!$ sequences from future consideration. The algorithm requires $(n-m)$ stages for finding the optimal solution. Some results on the performance of this algorithm are given in the next section.

NUMERICAL RESULTS

We will now present a simple example to illustrate how the algorithm proceeds from stage to stage. We will consider a problem with 4 jobs and 2 processors. The data required for the problem is given below in Tables 1 through 5.

TABLE 1.

Job Number i	Available Time (hrs) $A(i)$	Processing Time (hrs) $p(i)$	Waiting Cost ($/hr) $q(i)$	Due Date (hrs) $D(i)$	Penalty Rate ($/hr) $r(i)$
1	5	15	0.3	25	0.5
2	0	15	0.4	30	0.7
3	3	5	0.7	15	0.6
4	4	20	0.5	30	0.4

Processor Cost (x) = $5/hr.

TABLE 2. *Stage 1*

S_2	a_{S_2}	1_{S_2}	ΣWg	Σdg	$h(S_2)$	$f(S_2)$	$g(S_2)$
1, 2	15	20	0	0	0	100	100
1, 3	8	20	0	0	0	100	100
1, 4	20	24	0	0	0	120	120
2, 3	8	15	0	0	0	75	75
2, 4	15	24	0	0	0	120	120
3, 4	8	24	0	0	0	120	120

TABLE 3. *Stage 2*

S_2	k_3	S_3	a_{S_3}	1_{S_3}	ΣWg	Σdg	$h(S_3)$	$f(S_3)$	$g(S_3)$
1, 2	3	1, 2, 3	20	20	8.4	3.0	11.4	100	111.4
	4	1, 2, 4	20	35	5.5	2.0	7.5	175	182.5
1, 3	2	1, 3, 2	20	23	3.2	0.0	3.2	115	118.2
	4	1, 3, 4	20	28	2.0	0.0	2.0	140	142.0
1, 4	2	1, 4, 2	24	35	8.0	3.5	11.5	175	186.5
	3	1, 4, 3	24	25	11.9	6.0	17.5	125	142.5
2, 3	4	2, 3, 4	15	28	2.0	0.0	2.0	140	142.0
	1	2, 3, 1	15	23	3.2	0.0	3.2	115	118.2
2, 4	3	2, 4, 3	20	24	8.4	3.0	11.4	120	131.4
	1	2, 4, 1	24	30	3.0	2.5	5.5	150	160.5
3, 4	2	3, 4, 2	23	24	3.2	0.0	3.2	120	123.2
	1	3, 4, 1	23	24	0.9	0.0	0.9	120	120.9

Collecting sequences obtained in Table 3 into groups we obtain the data in Table 4.

TABLE 4.

S_3	a_{S_3}	$g(S_3)$	Eliminate	Retain
1, 2, 3	20	111. 4		X
1, 3, 2	20	118. 2	X	
2, 3, 1	15	118. 2		X
1, 2, 4	20	182. 5		X
1, 4, 2	24	186. 5	X	
2, 4, 1	24	160. 5		X
1, 3, 4	20	142. 0		X
1, 4, 3	24	142. 5	X	
3, 4, 1	23	120. 9		X
2, 3, 4	15	142. 0		X
2, 4, 3	20	131. 4		X
3, 4, 2	23	123. 2		X

TABLE 5. *Stage 3*

S_3	k_4	S_4	a_{S_4}	1_{S_1}	ΣWg	Σdg	$h(S_4)$	$f(S_4)$	$g(S_4)$
1, 2, 3	4	1, 2, 3, 4	20	40	16. 4	7. 0	23. 4	200	223. 4
1, 2, 4	3	1, 2, 4, 3	25	35	17. 4	8. 0	25. 4	175	200. 4
1, 3, 4	2	1, 3, 4, 2	28	35	28. 0	3. 5	31. 5	175	206. 5
2, 3, 4	1	*2, 3, 4, 1*	28	30	5. 0	2. 5	7. 5	150	**157. 5*
2, 3, 1	4	2, 3, 1, 4	23	35	6. 4	2. 0	8. 4	175	183. 4
2, 4, 3	1	2, 4, 3, 1	24	35	12. 9	8. 0	20. 0	175	195. 9
2, 4, 1	3	2, 4, 1, 3	29	30	17. 7	9. 9	27. 6	150	177. 6
3, 4, 2	1	3, 4, 2, 1	24	38	8. 6	6. 5	15. 1	190	205. 1
3, 4, 1	2	3, 4, 1, 2	24	38	10. 1	5. 6	15. 7	190	205. 7

The sequence with the smallest value of $g(S_4)$ is 2–3–4–1 (denoted by *). Complete enumeration also indicates that the sequence 2–3–4–1 is the minimum cost sequence.

The performance of the algorithm as compared to enumeration (which is the only other way to find the optimal sequence for the problem described here) depends upon the number of jobs, number of processors and the various cost and other data needed for the problem. To evaluate the performance of the algorithm, a number of simulation experiments were performed for different job-processor combinations. The data required for the problems were generated as uniformly distributed random numbers. The details are given below.

(a) Available Time—one digit number
(b) Processing Time—two digit number
(c) Waiting Cost—one digit number
(d) Due Date—sum of two, two-digit numbers
(e) Penalty Rate—one digit number
(f) Processor Cost—one digit number

Table 6 shows the mean and standard deviation of computation times (in 200 microseconds) for the algorithm. Table 7 shows the same data for complete enumeration. Figure 1 shows a plot of $n \times m$ against the difference in the computation times (Algorithm Time—Enumeration Time) as a percentage of the algorithm time. The simulation was performed on UNIVAC 1108 (EXEC–8) computer using a time-sharing terminal.

TABLE 6. *Data on the computation times (one unit=200 microsecs) for the algorithm*

[Mean/std. Deviation]

Processors (m)	Jobs (n)						
	3	4	5	6	7	8	9
2	0. 4/0. 2	3. 0/0. 62	13. 2/1. 55	47. 4/5. 65	161. 7/25. 9	450. 3/47. 77	1488. 5/190. 3
3		0. 7/0. 36	6. 1/0. 87	29. 0/3. 9	114. 5/12. 8	378. 7/55. 4	1071. 5/200. 8
4			1. 2/0. 33	10. 1/1. 05	65. 7/11. 1	251. 6/22. 67	876. 7/ 92. 3
5				1. 7/0. 44	17. 4/ 1. 5	92. 0/12/16	563. 0/ 84. 3

TABLE 7. *Data on the computation times (one unit=200 microsecs) for enumeration*

[Mean/Std. Deviation]

Processors (m)	Jobs (n)						
	3	4	5	6	7	8	9
2	0. 4/0. 1	1. 9/0. 61	11. 6/0. 72	84. 2/6. 77	727. 7/54. 9	*	*
3		1. 1/0. 0	7. 6/0. 7	59. 9/2. 51	512. 4/12. 52	*	*
4			4. 2/0. 22	38. 3/1. 99	439. 3/30. 8	*	*
5				20. 6/1. 37	234. 0/ 6. 99	*	*

(*) No data due to excessive computer time required.

The Tables 6 and 7 show that generally the algorithm does better than the enumeration for a large number of jobs and when (n-m) is large. The Figure 1 shows that the algorithm is definitely superior when ($n \times m$) is large.

CONCLUSION

An algorithm was proposed for minimizing total cost in parallel processing. The algorithm was described and a proof was given. The results of simulation indicated that algorithm performed better when ($n \times m$) is large (where, n=number of jobs, m=number of processors). As there exists no published algorithm for the problem proposed here, the performance of the algorithm could be compared only to complete enumeration. Further research in this area should consider the case of dynamic scheduling, where the number of jobs is not known initially. Another situation of equal importance is the case where the available times and processing times are probabilistic until the job actually starts processing. There are many variations of parallel processing that may be of particular interest in many practical applications.

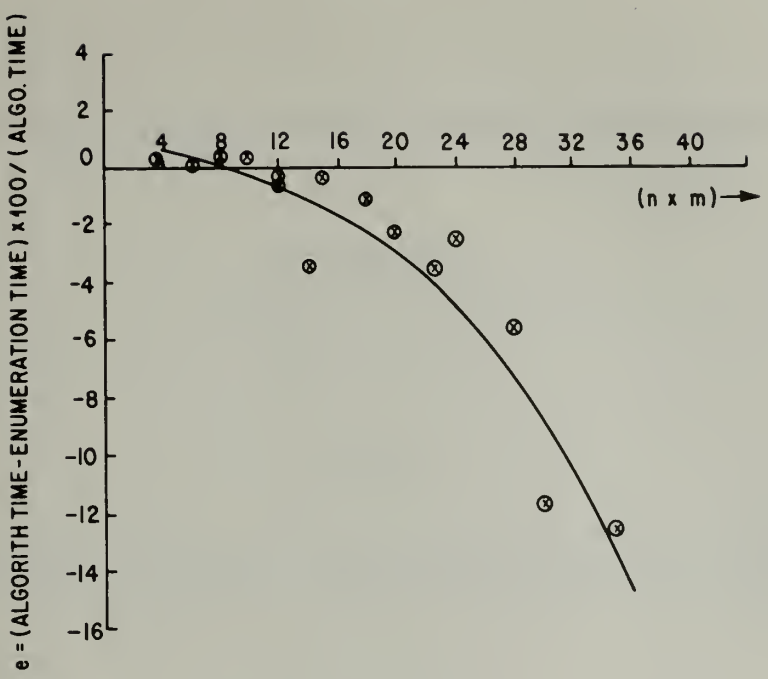


FIGURE 1.

REFERENCES

[1] Gupta, J. N. D. and A. R. Maykut, "Scheduling Jobs on Parallel Processors with Dynamic Programming," Decision Sciences, *4* (4) (1973).

[2] Held, M. and R. Karp, "A Dynamic Programming Approach to Sequencing Problems," Journal *10* (1962).

[3] Lawler, Eugene L., "On Scheduling Problems with Deferral Costs," Management Science, *11* (2) (1964).

[4] McNaughton, Robert, "Scheduling with Deadlines and Loss Functions," Management Science, *6* (1959).

[5] Rajaraman, M. K., "An Algorithm for Scheduling Parallel Processors," International Journal of Production Research, *13* (5) (1975).

[6] Rothkopf, Michael H., "Scheduling Independent Tasks on Parallel Processors," Management Science, *12* (5) (1966).

SPECIAL CASES OF THE FLOW-SHOP PROBLEM

Wlodzimierz Szwarc

University of Wisconsin-Milwaukee
Milwaukee, Wisconsin

ABSTRACT

The paper examines all known special cases of the $m \times n$ flow-shop problem. It provides solution procedures to three new special cases along with the optimality proofs. The theory of the new special cases is based on the critical path concept.

1. INTRODUCTION

Consider the $m \times n$ flow-shop problem which can be defined as follows. There are given n jobs $1, 2, \ldots, n$ each to be processed on m machines M_j, $j=1, \ldots, m$ in the same order $M_1, M_2, \ldots, M_m$. The processing order is assumed to be the same on each machine. Given the processing times of each job on each machine, the problem is to find an operation schedule for each machine so as to minimize the completion time. We will use the following notations:

$I=(1, 2, \ldots, n)$—the set of all the jobs.

$p=(p_1, p_2, \ldots, p_n)$—a permutation of the jobs of I.

t_{ij}—the processing time of job i on machine M_j.

$t_i \stackrel{df}{=} (t_{i1}, t_{i2}, \ldots, t_{im})$.

$t_j \stackrel{df}{=} (t_{1j}, t_{2j}, \ldots, t_{nj})$.

$t(p)$—the completion time of sequence p.

$g(p)$—the waiting time of machine M_m processing sequence p.

To simplify the notations we will usually assume $p=1, 2, \ldots, n$. Extending S. M. Johnson's findings ([2] p. 68) to the $m \times n$ case we have

$$t(p)=\max_{v_1, v_2, \ldots, v_{m-1}} \left(\sum_{i=1}^{v_1} t_{i1} + \sum_{i=v_1}^{v_2} t_{i2} + \ldots + \sum_{i=v_{m-1}}^{n} t_{im} \right) \tag{1}$$

where $v_1, v_2, \ldots, v_{m-1}$ are integers which satisfy the inequalities

$$1 \leq v_1 \leq v_2 \leq \ldots \leq v_{m-1} \leq n \tag{2}$$

Introduce

$$H_v^j = \sum_{i=1}^{v} t_{ij} - \sum_{i=1}^{v-1} t_{ij+1}, \quad j=1, \ldots, m-1 \tag{3}$$

Then $g(p)$, which is equal to $t(p)-\sum_{i=1}^{n} t_{im}$ can be presented in the following form:

$$g(p)=\max_{v_1, v_2, \ldots, v_{m-1}} (H_{v_1}{}^1+H_{v_2}{}^2+\ldots+H_{v_{m-1}}^{m-1}) \tag{4}$$

where the $v_1, v_2, \ldots, v_{m-1}$ satisfy (2).

Let Γ be a set of $m+n-1$ cells (i, j), $1\leq i\leq n$, $1\leq j\leq m$. We say that Γ is a path if one can arrange its elements in the following sequence:

$$\Gamma=(1, 1)\ldots(v_1, 1)(v_1, 2)\ldots(v_2, 2)\ldots(v_{m-1}, m)\ldots(n, m) \tag{5}$$

where the $v_1, v_2, \ldots, v_{m-1}$ are arbitrary integers satisfying (2).

REMARK: Any set of basic cells of an initial basic solution of a transportation problem obtained by the North-West Corner Rule could be viewed as a path.

The completion time $t(p)$ will be presented as

$$t(p)=\max_{\Gamma} \sum_{(i, j)\epsilon\Gamma} t_{ij}$$

Set Γ is a critical path if

$$t(p)=\sum_{(i, j)\epsilon\Gamma} t_{ij}$$

The following special cases have been considered:

CASE 1 [1]: There exists an integer $1\leq h\leq m-1$ such that

$\min_i t_{ij}\geq\max_i t_{ij+1}$, $j=1, 2, \ldots, h-1$; and

$\max_i t_{ij}\leq\min_i t_{ij+1}$, $j=h+1, \ldots, m-1$.

CASE 2 [1]:

Case 2a: $\max_i t_{ij}\leq\min_i t_{ij+1}$, $j=1, 2, \ldots, m-2$.

Case 2b: $\min_i t_{ij}\geq\max_i t_{ij+1}$, $j=2, \ldots, m-1$.

CASE 3 [3]: For each $i_1, i_2 \epsilon I$ and $j_1, j_2\leq m$,

$$t_{i_1}\leq t_{i_2} \text{ or } t_{i_1}\geq t_{i_2} \tag{6}$$

and

$$t^{j_1}\leq t^{j_2} \text{ or } t^{j_1}\geq t^{j_2}. \tag{7}$$

Case 3a: $t^1\geq t^j$ and (6), (7), $j=2, \ldots, m$,

Case 3b: $t^j\leq t^m$ and (6), (7), $j=2, \ldots, m$.

CASE 3′ [1]:

Case 3′a: $t^1\geq t^j$ and (6), $j=2, \ldots, m$,

Case 3′b: $t^j\leq t^m$ and (6), $j=2, \ldots, m$.*

*The assumption made in [1] about the existence of a sequence S is redundant since it follows from (6).

J. N. D. Gupta [1] developed a comprehensive theory for Cases 1, 2, and 3′. His results are based on Theorem 2—a simplified version of W. E. Smith's theorem ([4] p. 60) which according to [1] p. 258 "provides the easiest and simplest approach to sequencing and scheduling problems." He uses the following known formula for the completion time $t(\sigma a, j)$ of sequence σa ($\sigma a \epsilon I$, $a \bar{\epsilon} \sigma$) on machines $M_1, M_2, \ldots, M_j, j=1, \ldots m$.

$$t(\sigma a, j)=\max [t(\sigma, j), t(\sigma a, j-1)]+t_{aj} \tag{8}$$

where a is the last job of sequence σa.

Section 2 demonstrates that Cases 1 and 2 can be reduced to Cases 1 and 2 of [5] where the three machine version has been considered. The solution of those cases is based on a simple observation already used in [2].† Due to the obvious identity

$$H_{v+1}^{j}=H_{v}^{j}+t_{i+1,j}-t_{v,j+1} \tag{9}$$

the following implications hold

$$\min_i t_{ij} \geq \max_i t_{ij+1} \Rightarrow H_v^j \leq H_{v+1}^j \text{ for every } v \tag{10}$$

$$\max_i t_{ij} \leq \min_i t_{ij+1} \Rightarrow H_v^j \geq H_{v+1}^j \text{ for every } v. \tag{11}$$

It is far more convenient to minimize $g(p)$ defined by (4) rather than $t(p)$ $(=g(p)+\text{constant})$ since properties (9) and (10) lead to the solution method almost immediately. The author of [1] developed seven lemmas and five theorems to prove those two cases.

Section 2 discusses the mathematical aspects of Case 3 based on the critical path concept as developed in [3]. To get a better insight we examined how each of the conditions (6) and (7) affects the properties of the critical path. This approach proved to be useful, since it enabled us to solve the following new special cases:

CASE 4:

$$t^j \leq t^1, t^j \leq t^m, j=2, \ldots, m-1, \text{ and (6)} \tag{12}$$

CASE 5:

$$m=3, t^2 \leq t^1, t^2 \leq t^3. \tag{13}$$

Section 3 demonstrates, using two examples, that the method of cases 3′a and 3′b of [1] does not always produce optimal solutions, contrary to what Theorems 7 and 6 of [1] claim.

Finally, we established a lower bound for the minimal $t(p)$.

Since this paper deals primarily with [1], [3], and [5] we assume the reader to be familiar with those references.

†Where $v_1=u$, $v_2=v$, $H_{v_1}{}^1=K_u$, $H_{v_2}{}^2=H_v$.

2. KNOWN SPECIAL CASES

CASE 1:

According to (9), (10), (11), the assumptions of this case imply

$$H_v{}^j \leq H_{v+1}^j,\ j=1, \ldots, h-1, \text{ and } H_v{}^j \geq H_{v+1}^j,\ j=h+1, \ldots, m-1.$$

Hence

$$g(p)=\max_{1\leq v_h\leq n}(H_{v_h}{}^1+H_{v_h}{}^2+\ldots+H_{v_h}^{m-1}).$$

Denote v_h by v. Then

$$g(p)=\max_{1\leq v\leq n}\left[\sum_{i=1}^{v}(t_{i1}+t_{i2}+\ldots+t_{im-1})-\sum_{i=1}^{v-1}(t_{i2}+t_{i3}+\ldots+t_{im})\right]. \tag{14}$$

Hence Case 1 has a two machine "structure."

Let

$$\left.\begin{aligned} A_i&=t_{i1}+\ldots+t_{im-1}\\ B_i&=t_{i2}+\ldots+t_{im}\end{aligned}\right\}. \tag{15}$$

To solve this case apply a method which will be later called *Johnson's Approximate Method:* Solve the $2\times n$ AB problem where the processing times A_i and B_i on "machines" A and B are defined by (15).*

CASE 2a: In view of (9), (10), and (11), and the assumptions: $H_v{}^j \geq H_v{}^j+1, j=1,2, \ldots,$ m-2.

Therefore

$$g(p)=\max_{1\leq v_{m-1}\leq n}(H_1{}^1+H_1{}^2+\ldots+H_1^{m-2}+H_{v_{m-1}}^{m-1})=\sum_{j=1}^{m-2}t_{1j}+\max_{1\leq v_{m-1}\leq n}H_{v_{m-1}}^{m-1}.$$

To find a sequence which minimizes $g(p)$ we proceed as follows:

1. Solve the $2\times n$ problem $M_{m-1}\ M_m$ (thereby minimizing max H_{vm-1}^{m-1}—the waiting time of machine M_m of the two machine problem $M_{m-1}\ M_m$). Let $p=(1, 2, \ldots, n)$ be the optimal sequence.

2. Find $J=\left\{i \;\middle|\; \sum_{j=1}^{m-2}t_{ij}<\sum_{j=1}^{m-2}t_{1j}\right\}$. Define $p^{(i)}=(i, 1, 2, \ldots, i-1, i+1, \ldots, n),\ i\epsilon J$.

3. Among p and the $p^{(i)}$ find a permutation with the smallest processing time. This sequence is the optimal solution.

REMARK: Denote $A_1=\sum_{j=1}^{m-2}t_{1j},\ v=v_{m-1},\ H_v=H_{vm-1}^{m-1}$. Then this case becomes Case 2a of [5] since both $g(p)$ are identical.

CASE 2b:

$$H_v^j \leq H_{v+1}^j,\ j=2, 3, \ldots, m-1.$$

Thus

$$g(p)=\max_{1\leq v_1\leq n}(H_{v_1}^1+H_n^2+H_n^3+\ldots+H_n^{m-1})=\sum_{i=1}^{n}\sum_{j=2}^{m-1}t_{ij}-\sum_{i=1}^{n-1}\sum_{j=3}^{m}t_{ij}+\max_{1\leq v_1\leq n}H_{v_1}^1.$$

Introduce $g'(p)=g(p)-\sum_{i=1}^{n}\sum_{j=2}^{m-1}t_{ij}+\sum_{i=1}^{n}\sum_{j=3}^{m}t_{ij}=g(p)+$ constant.

---

*The author of [1] solved several two-machine problems using the $f(i)$ function. This ordering is equivalent to Mitten's rule ([5], p. 146).

Then

$$g'(p)=\sum_{j=3}^{m} t_{nj}+\max_{1\le v_1\le n} H_{v_1}^1$$

SOLUTION: 1. Solve the $2\times n$ problem M_1M_2 (thereby minimizing max $H_{v_1}^1$—the waiting time of machine M_2 of the M_1M_2 problem). Let $p=(1, 2, \ldots, n)$ be the optimal sequence.

2. Find $J\overset{df}{=}\left\{i \;\middle|\; \sum_{j=3}^{m} t_{ij}<\sum_{j=3}^{m} t_{nj}\right\}$. Define $p^{(i)}=(1, 2, \ldots, i-1, i+1, \ldots, n, i)$, $i\epsilon J$.

3. Among p and the $p^{(i)}$ find a permutation with the smallest processing time. This is the optimal solution.

REMARK: Denote $C_n=\sum_{j=3}^{m} t_{nj}$, $u=v_1$, and $K_u=H_{v_1}^1$. Then this case becomes Case 2b of [5] since the $g'(p)$ are identical.

CASE 3a: Introduce the following two conditions:

(16) For every s, $s\le m-1$: $t_{v_s}\ge t_i$ for each $i>v_s$.

(17) If $v_s\neq v_{s+1}$ then $t^s\ge t^j$ for each $j>s$.

Under the assumptions $t^1\ge t^j$, $j=2, \ldots m$ the following three results hold.

THEOREM 1. Condition (6) implies that for every sequence p there exists a critical path which satisfies (16).

THEOREM 2. Conditions (6) and (7) imply that for every sequence p there exists acritical path which satisfies (16) and (17).

THEOREM 3. Consider two sequences $p=(1, 2, \ldots, n)$ and $p'=(1, 2, \ldots, k\text{-}1, k+1, k, k+2, \ldots, n)$. Then (6) and (7) imply

$$(t_k\ge t_{k+1})\Longrightarrow(t(p)\le t(p'))$$

The proofs of those results can be found in [3].*

From Theorem 3 follows

COROLLARY 1. Permutation $p=(1,2, \ldots, n)$ is optimal if $t_i\ge t_{i+1}$ for each i.

CASE 3b: This case is symmetrical to the case 3a. Permutation $p=(1,2, \ldots, n)$ is optimal if $t_i\le t_{i+1}$ for each i.†

3. SOME REMARKS ON CASE 3′

Case 3′, which is a generalization of Case 3, was first considered by J. N. D. Gupta [1]. According to Theorems 7 and 6 of [1] sequence $p=(1, 2, \ldots, n)$ is optimal if

(1) $t_i\ge t_{i+1}$ for each i (Case 3′a).

(2) $t_i\ge t_{i+1}$ for each i (Case 3′b).

*Theorem 3 relies on Theorem 2 exclusively. However, to prove Theorem 2, one has to consider only those paths which satisfy condition (16) of Theorem 1.

†Conditions (16) and (17) become in this case:

$$t_{v_s}\ge t_i,\ i<v_s.$$

$$v_s\neq v_{s+1}\Longrightarrow t^s\ge t^j,\ j<s.$$

Introduce the following two examples.

EXAMPLE 1 (Case 3'a)

	M_1	M_2	M_3
1	9^9	8^{17}	4^{21}
2	6^{15}	5^{22}	4^{26}
3	3^{18}	2^{24}	3^{29}

	M_1	M_2	M_3
1	9^9	8^{17}	4^{21}
3	3^{12}	2^{19}	3^{24}
2	6^{18}	5^{24}	4^{28}

REMARK: The numbers above the processing times denote completion times.

Sequence 132 is optimal ($t(132)=28$) whereas sequence 123 which satisfies the conditions of Theorem 7 is not ($t(123)=29$).

EXAMPLE 2 (Case 3'b)

	M_1	M_2	M_3
1	3^3	2^5	3^8
2	4^7	5^{12}	6^{18}
3	4^{11}	8^{20}	9^{29}

	M_1	M_2	M_3
2	4^4	5^9	6^{15}
1	3^7	2^{11}	3^{18}
3	4^{11}	8^{19}	9^{28}

Again it is 213 that is optimal and not sequence 123 which satisfies the conditions of Theorem 6.

Let us examine Example 2. It is easy to see that only $t_{12}=2$ causes violations of condition (7) since neither $t^1 \geq t^2$ nor $t^1 \leq t^2$ hold. Inequality $t(213)<t(123)$ remains true as long as $t_{12}=3-\epsilon$, $\epsilon>0$ no matter how small ϵ is. For $t_{12}=3$ condition (6) holds and sequence 123 is optimal. Then we deal with Case 3b, however.*

*The reason why the proof of Theorem 6 ([1] p. 267) failed is due to the fact that (44) should be an $\geq$ inequality and not an equation in view of (*).

$$\sum_{s=m}^{u} t_{bs} + \sum_{s=u}^{M} t_{cs} \geq \max\left[t_{bm} + \sum_{s=m}^{M} t_{cs},\ \sum_{s=m}^{M} t_{bs} + t_{cM} \right] \qquad (*)$$

Let me demonstrate this using example 2. Assume $\sigma=\phi$, $a=1$, $b=2$, $c=3$, $M=3$, $1 \leq m \leq 3$. Then equation (44) becomes $T(abc, 3)=t(123)=29 \neq \max(3+4+21, 5+5+17, 8+6+9; 18+9)=28$.

One can find a similar deficiency in the proof of Lemma 8 (p. 266). Instead of replacing

$$\sum_{s=m}^{u} t_{as} + \sum_{s=u}^{M} t_{bs} \text{ directly by } t_{aM} + \sum_{s=m}^{M} t_{bs}$$

the author of [1] replaced it first by $t_{aM}+\Sigma t_{bs}$. Hence the intermediate expression of $T(\sigma ab, M)$ is not correct. This, however, did not affect the validity of the lemma 8 since t_{am} had been eventually replaced by t_{aM}.

4. NEW SPECIAL CASES

Case 4:

(12) $$t^j \le t^1,\ t^j \le t^m,\ j=2, \ldots, m-1, \text{ and } (6)$$

The following theorem holds.

THEOREM 4. For every sequence p conditions (12) imply an existence of a critical path Γ where $v_1=v_2= \ldots =v_{m-1}$.

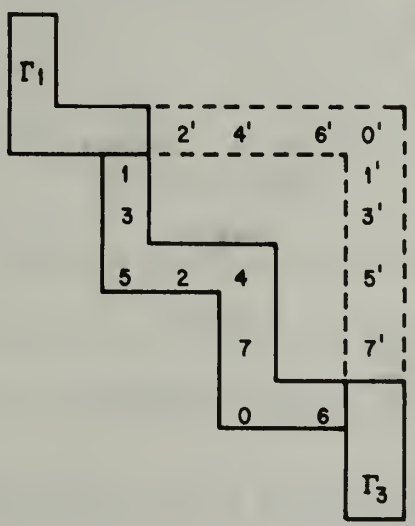
PROOF: Let $p=(1, 2, \ldots, n)$. In view of (6) and Theorem 1 we will consider only such sets Γ which satisfy (16). To prove the theorem one has to show that to any such Γ there exists a path

$$\Gamma'=(1, 1) \ldots (v_1, 1)(v_1, 2) \ldots (v_1, m)(v_1+1, m) \ldots (n, m)$$

that

(18) $$\sum_{(i,j)\in\Gamma} t_{ij} \le \sum_{(i,j)\in\Gamma'} t_{ij}$$

We will demonstrate the proof on the following scheme:



Let Γ_2 and Γ'_2 be sets of marked cells (by digits) within the solid and dotted lines respectively.

Define

$\Gamma=\Gamma_1\Gamma_2\Gamma_3$ and $\Gamma'=\Gamma_1\Gamma'_2\Gamma_3$. To prove (18) it is sufficient to show

(18′) $$\sum_{\Gamma_2} t_{ij} \le \sum_{\Gamma'_2} t_{ij}$$

By x_k and y_k denote the t_{ij} of the cells (i, j) marked by digits k and k' respectively. Then

$x_k \le y_k$, $k=1, 3, \ldots$, by (12),

$x_k \le y_k$, $k=2, 4, \ldots$, by (16).

Since $x_0 \le y_4 \le y_0$ (by (16) and (12)) we have

$x_k \le y_k$ for all k.

Hence (18′) is satisfied and so is (18). Q.E.D.

Case 5 is solvable by Johnson's Approximate Method since $g(p)$ is equal to the right-hand side of (14) (in view of Theorem 4).

CASE 5:

(13) $$m=3,\ t^2\leq t^1,\ t^2\leq t^3$$

THEOREM 5. For every sequence p conditions (13) imply an existence of a critical path where $v_1=v_2$.

PROOF: Assume $p=(1, 2, \ldots, n)$, and let

$$\Gamma=(1, 1)\ldots(v_1, 1)(v_1, 2)\ldots(v_2, 2)(v_2, 3)\ldots(n, 3).$$

We have to show that to each such Γ there exists a Γ' with $v_1=v_2$ so that (18) holds. There are two possible cases.

1. $t_{v_1 2}\leq t_{v_2 1}$.

Define

$$\Gamma'=(1, 1)\ldots(v_1, 1)(v_1+1, 1)\ldots(v_2, 1)(v_2, 2)(v_2, 3)\ldots(n, 3).$$

The assumption, and (13) imply (18). Q.E.D.

2. $t_{v_1 2}>t_{v_2 1}$.

We have

$$t_{v_2 2}\leq t_{v_2 1}\Longrightarrow t_{v_2 2}<t_{v_1 2}.$$

From

$$t_{v_1 2}\leq t_{v_1 3}$$

follows

(19) $$t_{v_2 2}<t_{v_1 3}.$$

Define

$$\Gamma'=(1, 1)\ldots(v_1, 1)(v_1, 2)(v_1, 3)(v_1+1, 3)\ldots(n, 3).$$

Again (13), and (19) imply (18). Q.E.D

CONCLUSION: According to Theorem 5, Case 5 can be solved by Johnson's Approximate Method.

* *
*

The reader may wonder whether Case 5 can be extended beyond $m=3$. Consider, for instance the following (quite restrictive) case:

(20) $$t^1\geq t^2\geq \ldots \geq t^h\leq t^{h+1}\leq \ldots \leq t^m,\ 2\leq h\leq m-1.$$

The following example shows that Theorem 5 does not hold.

10^{10}	9^{19}	1^{20}	9^{29}
8^{18}	7^{26}	4^{30}	7^{37}

Here condition (20) is satisfied for $m=4$ and $h=3$. However, the only critical path $\Gamma=(1, 1)$ $(1, 2)$ $(2, 2)$ $(2, 3)$ $(2, 4)$ does not satisfy the condition of Theorem 5 since $v_1=1\neq v_2=2$. Hence Johnson's Approximate Method does not guarantee optimality in this case.

5. SUFFICIENT OPTIMALITY CONDITIONS FOR JOHNSON'S APPROXIMATE METHOD

For a given $m\times n$ problem define a respective $2\times n$ time lag problem where the items are operated on machines M_1 and M_m and where the $\sum_{j=2}^{m-1} t_{ij}$ is the waiting time for item i after it had been processed on machine M_1 and before it starts on machine M_m. The processing order is M_1M_m for all the items. We also assume that the processing order (to be found) is the same on both machines.

The $2\times n$ time lag problem is *less restrictive* than the $m\times n$ problem since the time lags Σt_{ij} and $\Sigma t_{i+1,j}$ may overlap whereas in the $m\times n$ they cannot. Hence

$$\min_p G(p)\leq \min_p g(p)$$

where $G(p)$ is the idle time of machine M_m of the $2\times n$ time lag problem.

Proceeding the same way as in [5],* p. 151, one can prove that

$$G(p)=\max_{1\leq v\leq n}\left[\sum_{i=1}^{v}(t_{i1}+\ldots+t_{im-1})-\sum_{i=1}^{v-1}(t_{i2}+\ldots+t_{im})\right]$$

which is identical with (14) (Case 1).

Hence we can define the sufficient optimality condition for Johnson's Approximate Method as follows:

Let p be the optimal solution of the $2\times n$ problem KL, where

$$K_i=\sum_{j=1}^{m-1} t_{ij},\quad L_i=\sum_{j=2}^{m} t_{ij}$$

are the processing times of item i on "machines" K and L.

By $T(p)$ denote the completion time of problem KL. Sequence p is the optimal solution of the $m\times n$ problem if

$$t(p)=T(p)-\sum_{j=2}^{m-1} t_{ij} \tag{21}$$

Remark: The right hand side of (21) is the lower bound for the minimal processing time.

One should stress that (21) is by no means a necessary optimality condition. For instance the solutions produced by Johnson's Approximate Method for Cases 3a and 3b, although always optimal, often do not satisfy (21), however.

---

*By making the following substitutions:

A_i by t_{i1}, B_i by $\sum_{j=2}^{m-1} t_{ij}$, C_i by t_{im}, $g(p)$ by $G(p)$, and u by v.

REFERENCES

[1] Gupta, J. N. D., "Optimal Schedules for Special Structure Flowshops," Naval Research Logistics Quarterly, *22*, 255–269 (1975).

[2] Johnson, S. M. "Optimal Two and Three-Stage Production Schedules with Setup Times Included," Naval Research Logistics Quarterly, *1*, 61–68 (1954).

[3] Smith, M. L., S. S. Panwalkar, and R. A. Dudek, "Flowshop Sequencing Problem with Ordered Processing Time Matrices," Management Science, *21*, 544–549 (1975).

[4] Smith, W. E. "Various Optimizers for Single-Stage Production," Naval Research Logistics Quarterly, *3*, 56–66 (1956).

[5] Szwarc, W. "Mathematical Aspects of the $3\times n$ Job-Shop Sequencing Problem," Naval Research Logistics Quarterly, *21*, 145–153 (1974); and "A Note on Mathematical Aspects of the $3\times n$ Job-Shop Sequencing Problem," Naval Research Logistics Quarterly, *21*, 725–726 (1974).

ADAPTIVE FORECASTING OF THE SIZE OF A WORK FORCE SUBJECT TO RANDOM WITHDRAWALS*

S. Zacks

Case Western Reserve University
Cleveland, Ohio

S. E. Haber

The George Washington University
Washington, D.C.

ABSTRACT

Adaptive forecasting procedures are developed for predicting the size of a work force which is subject to random withdrawals. The procedures are illustrated using Marine Corps data in which four stages of service for incoming cohorts are distinguished. Using these data, three forecasting procedures—conditional maximum likelihood estimation of prediction intervals; tolerance intervals; and Bayes prediction intervals—are compared.

1. INTRODUCTION

In a recent technical paper [4] a method of forecasting the size of a force subject to random withdrawals was provided. The method presented there considers the size of a force at six months intervals, estimates the retention rates of certain subgroups (cohorts) at the time of forecasting, and provides a prediction interval for the anticipated size of the force six or twelve months ahead. The method developed in [4] was specifically oriented to the problem of forecasting the number of first-term enlistees in the Marine Corps. In [5] and [6] it was applied to the problem of estimating the total size of the Marine Corps.

We distinguish four phases in the service of an enlistee. Phase I consists of the first six months of basic training; Phase II follows Phase I and continues until six months before the termination of the (first) contract. Phase III consists of the last six months of the (first) contract. Phase IV is the phase of career service. We have shown in [4] that the first three phases are similar to those of reliability systems in which the failure rate function attains, after an initially high level, stable failure level (Phase II), and then reaches a phase of wearout (Phase III).

* Supported by the Office of Naval Research under Contract N00014-75-C-0729, Project NR 347-020 at the Program in Logistics, The George Washington University and under Contract N00014-75-C-0529, Project 042-276 at Case Western Reserve University

In the present paper we develop the theory for Phase II forecasting and compare three alternative procedures. One procedure provides conditional maximum likelihood estimates (CMLE) of the limits of prediction intervals. The second procedure determines tolerance intervals; the third procedure provides Bayes prediction intervals for the size of the force. We show that although the CMLE procedure is less conservative than the other two procedures, it has yielded prediction intervals for the total force which contained the actual values (on a retroactive basis). This is due to the fact that the samples are large and the standard errors of the estimates are sufficiently small.

The Phase II forecasting is performed for cohorts consisting of enlistees classified according to the following factors: time of entry, length of contract, race and education. Haber has shown in [2] that among many factors that can possibly affect retention rates, the above factors are the most significant. We base the model of the present study on the same factors. The structure of the data as related to these factors is explained in Section 2. In Section 3 we discuss the basic statistical model for adaptive forecasting. In Section 4 we develop the concept of prediction intervals as a tool of forecasting. In Section 5 we discuss CMLE-prediction limits, tolerance limits and *Bayes* prediction limits for cohorts in Phase II more than six months (i.e., at least 12 months in service). The forecasting procedures are compared numerically by applying them retroactively to actual Marine Corps data for the period January 1969 to July 1972. Section 6 is devoted to the forecasting problem for cohorts just entering Phase II. The methods developed are related to those of Section 5. FORTRAN programs used in the study are available upon request. Finally, although there is a vast literature on subjects touched on or mentioned in the present study, either from the point of view of manpower statistics, reliability theory, Poisson processes, statistical distributions, or Bayesian analysis, we site only those references that are most relevant to the discussion.

2. THE DATA AND STRUCTURAL COMPONENTS

The random variables considered are the number of enlistees remaining in service at six months intervals, in groups (cohorts) classified according to the following characteristics:

A. Length of enlistment contract (LC):
- Two years, $i=1$
- Three years, $i=2$
- Four years, $i=3$

B. Education Level (ED):
- Less than high school, $j=1$ (LHS)
- High school or more, $j=2$ (AHS)

C. Race (R):
- White, $k=1$ (W)
- Non-White, $k=2$ (NW)

D. Time of Entry into Service (in six months periods) (EN):
- January 1–June 30, 1969 $l=1$ (1–69)
- July 1–December 31, 1969 $l=2$ (7–69)
- Up to January 1–June 30, 1972 $l=7$ (1–72)

TABLE 1. *Number of enlistees remaining in service in periods starting on July 1, 1971, January 1, 1972 and July 1, 1972*

LC	ED	R	EN	7–1–71	1–1–72	7–1–72
2	LHS	W	1–71	1564	1489	1418
2	LHS	NW	1–71	537	525	499
2	AHS	W	1–71	3416	3351	3280
2	AHS	NW	1–71	720	702	674
2	LHS	W	7–71	2362	1809	1717
2	LHS	NW	7–71	916	744	716
2	AHS	W	7–71	4905	4553	4451
2	AHS	NW	7–71	1353	1235	1195
3	LHS	W	1–70	1262	1099	1009
3	LHS	NW	1–70	173	145	135
3	AHS	W	1–70	2171	2079	2019
3	AHS	NW	1–70	282	267	256
3	LHS	W	7–70	1143	1012	910
3	LHS	NW	7–70	131	117	104
3	AHS	W	7–70	2437	2364	2301
3	AHS	NW	7–70	251	240	230
3	LHS	W	1–71	1293	1219	1155
3	LHS	NW	1–71	181	172	163
3	AHS	W	1–71	1180	1151	1121
3	AHS	NW	1–71	190	183	177
3	LHS	W	7–71	1102	919	866
3	LHS	NW	7–71	170	143	133
3	AHS	W	7–71	1115	1034	997
3	AHS	NW	7–71	260	233	227
4	LHS	W	1–69	1909	1688	1570
4	LHS	NW	1–69	221	186	165
4	AHS	W	1–69	3652	3501	3408
4	AHS	NW	1–69	382	360	345
4	LHS	W	7–69	2226	1957	1781
4	LHS	NW	7–69	244	208	190
4	AHS	W	7–69	5040	4841	4675
4	AHS	NW	7–69	507	473	451
4	LHS	W	1–70	2935	2594	2347
4	LHS	NW	1–70	384	327	292
4	AHS	W	1–70	3715	3556	3416
4	AHS	NW	1–70	489	454	428
4	LHS	W	7–70	3173	2864	2633
4	LHS	NW	7–70	356	321	294
4	AHS	W	7–70	4401	4267	4108
4	AHS	NW	7–70	547	511	478
4	LHS	W	1–71	3325	3146	2979
4	LHS	NW	1–71	423	410	386
4	AHS	W	1–71	3366	3294	3207
4	AHS	NW	1–71	559	546	526
4	LHS	W	7–71	3969	3358	3174
4	LHS	NW	7–71	479	402	383
4	AHS	W	7–71	4600	4445	4281
4	AHS	NW	7–71	688	632	607

In Table 1 we present the observed values of these random variables at the time intervals starting on July 1, 1971, January 1, 1972 and July 1, 1972. These values are later used to illustrate the forecasting procedures.

As in the previous study [4], we distinguish between three phases in the service of first-term enlistees.

Phase I: the first six months of basic training.

Phase II: the period following Phase I up to the last six months before termination of contract.

Phase III: the last six months of the contract.

In Figure 1 we illustrate the retention of cohorts (in percents) in relation to their three phases of service.

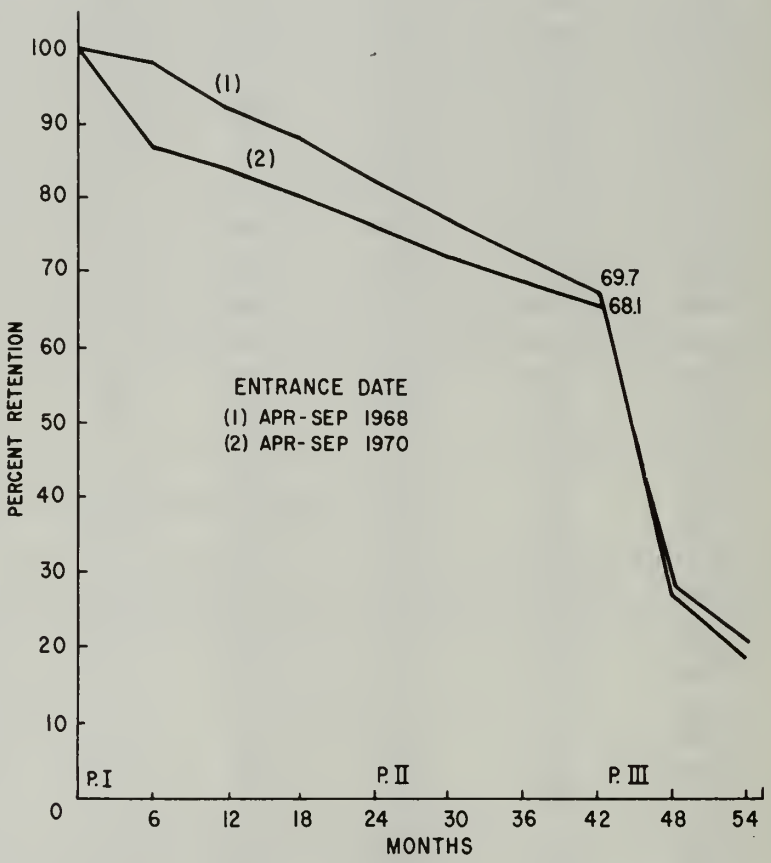


FIGURE 1. Retention of first enlistees (in percents) among four-year, non-white, above high school cohorts.

3. THE STATISTICAL MODEL

Let $X_{ijkl}(t)$ denote the number of enlistees remaining in service at the end of the *t-th* epoc where an epoch is a six-month period, from the group (cohort) defined by (i, j, k, ℓ) combination The statistical model specifies that the conditional distribution of $X_{ijkl}(t+1)$ given $X_{ijkl}(t)$ the *binomial* distribution with parameters $X_{ijkl}(t)$ (number of Bernoulli trails) and $\theta_{ijkl}(t+$ (retention probability) i.e., for every $t \geq 1$, and all (i, j, k, ℓ)

$$X_{ijkl}(t+1)|X_{ijkl}(t)\sim B(X_{ijkl}(t), \theta_{ijkl}(t+1)). \tag{3.1}$$

Generally $B(N, p)$ denotes a binomial distribution corresponding to N Bernoulli trials and probability of success p, $0<p<1$. The symbol $\sim$ in (3.1) designates that, conditionally on $X_{ijkl}(t)$, the random variable $X_{ijkl}(t+1)$ has the specified binomial distribution. Indeed, the number of enlistees from the (i, j, k, l) cohort remaining in service at the $(t+1)$st epoch can assume any value from 0 to $X_{ijkl}(t)$. It is assumed that each individual in a given cohort has the same retention probability in a given future period and that the events of retention or withdrawal of different individuals are independent. This model may be justifiably criticized if applied to small groups, due to possible interactive forces among the members. It seems, however, to be an adequate model for describing retention in large cohorts.

4. THE PREDICTION INTERVALS

After observing the sequence of X-values for each (i, j, k, l) combination, at epochs 1, . . ., t, we wish to forecast their possible values s epochs in the future. One can easily verify that the conditional distribution of $X_{ijkl}(t+s)$, given $X_{ijkl}(t)$ is the binomial $B(X_{ijkl}(t), P_{ijkl}(t, s))$ where,

$$P_{ijkl}(t, s)=\prod_{v=1}^{s} \theta_{ijkl}(t+v),\ t=0, 1, \ldots, s=1, 2, \ldots. \tag{4.1}$$

Accordingly, if the sequence of retention probabilities $\underset{\sim}{\theta}_{ijkl}=\{\theta_{ijkl}(t);\ t\geq 1\}$ is known one can determine for each t and s prediction intervals $(\underline{\eta}_{ijkl}(t+s), \overline{\eta}_{ijkl}(t+s))$, satisfying the requirement that, for each $t\geq 0$

$$P[\underline{\eta}_{ijkl}(t+s)\leq X_{ijkl}(t+s)\leq \overline{\eta}_{ikjl}(t+s)|X_{ijkl}(t), \underset{\sim}{\theta}_{ijkl}\}\geq 1-\alpha \tag{4.2}$$

$(1-\alpha)$ is a preassigned probability level.

The limits of these prediction intervals depend on $X_{ijkl}(t)$ and on $P_{ijkl}(t, s)$ and are given by

$$\underline{\eta}_{ijkl}(t+s)=\underline{\text{maximal non-negative integer, } k,}$$

$$\underline{\text{such that}}\ B(k|X_{ijkl}(t), P_{ijkl}(t,s))\leq\frac{\alpha}{2} \tag{4.3}$$

and

$$\overline{\eta}_{ijkl}(t+s)=\underline{\text{least non-negative integer, } k,}$$

$$\underline{\text{such that}}\ B(k|X_{ijkl}(t), P_{ijkl}(t, s))\geq 1-\frac{\alpha}{2}; \tag{4.4}$$

where $B(k|X_{ijkl}(t), P_{ijkl}(t, s))$ is the c.d.f. of the corresponding binomial distribution. The determination of prediction limits (4.3) and (4.4) requires the computation of the corresponding binomial c.d.f.'s. In the cases considered here, the X-values are generally in the order of several hundreds or thousands (see Table 1). In these cases the numerical determination of the binomial c.d.f. is given by the normal approximation (see Johnson and Kotz [3], p. 62). Based on this approximation we employ the following large sample formulae for the prediction limits

$$\bar{\eta}_{ijkl}(t+s)=X_{ijkl}(t)P_{ijkl}(t,s)+z_{1-\alpha/2}\{X_{ijkl}(t)\cdot P_{ijkl}(t,s)\cdot Q_{ijkl}(t,s)\}^{1/2}$$

and

$$\underline{\eta}_{ijkl}(t+s)=X_{ijkl}(t)P_{ijkl}(t,s)-z_{1-\alpha/2}\{X_{ijkl}(t)P_{ijkl}(t,s)Q_{ijkl}(t,s)\}^{1/2}$$

where $Q_{ijkl}(t,s)=1-P_{ijkl}(t,s)$ and $z_{1-\alpha/2}$ is the $(1-\alpha/2)$th fractile of the standard normal distribution. Generally the values of the retention probabilities $P_{ijkl}(t,s)$ are unknown and should be estimated from the data. In the following sections we discuss the problem of determining the prediction limit when the retention probabilities are unknown.

5. ESTIMATING THE PREDICTION INTERVALS

In the present section we discuss three methods of estimating the prediction intervals when the retention probabilities are unknown. For the sake of simplifying the presentations we restrict attention to the problem of forecasting force size six months ahead, i.e., $s=1$. We discuss here only cohorts in Phase II, which have been in service at least 12 months, i.e., $t\geq 2$. The case $t=1$ will be discussed later.

5.1 CMLE-Prediction Limits

For estimating the retention probabilities $\theta_{ijkl}(t+1), t\geq 2$, we develop methods which depend only on the observed statistics of the cohort under consideration. These methods can be modified to utilize information for previous cohorts (as will be indicated in Section 6). For this purpose we have to introduce the assumption that all the cohorts having the same length of contract and the same education and race characteristics behave similarly with respect to their attrition, irrespective of the period of entry into service. The methods discussed in the present section are free of such an assumption. We have to assume, however, that the retention probabilities of each cohort have close values in adjacent periods. Thus, we can estimate $\theta_{ijkl}(t)$ by the *conditional maximum likelihood* (CMLE) estimates

$$\hat{\theta}_{ijkl}(t)=X_{ijkl}(t)/X_{ijkl}(t-1),\ t\geq 2. \tag{5.1}$$

This estimator employs, for each cohort, only the current and the previous six months values of X. Point estimates of the prediction limits for $X_{ijkl}(t+1)$ are obtained by substituting in formula (4.3) the values of $\hat{\theta}_{ijkl}(t)$ for $P_{ijkl}(t,s)$. These estimates of the prediction limits are labelled the *CMLE-prediction limits*. They are given by the formula

$$X_{ijkl}(t)\cdot\hat{\theta}_{ijkl}(t)\pm z_{1-\alpha/2}\{X_{ijkl}(t)\hat{\theta}_{ijkl}(t)\cdot(1-\hat{\theta}_{ijkl}(t))\}^{1/2}. \tag{5.2}$$

Indeed, according to the invariance principle of maximum likelihood estimation (see Zacks [7], p. 223) the CMLE limits given by (5.2) are themselves conditional maximum likelihood estimators of the prediction limits. Furthermore, according to the large sample theory of maximum likelihood estimators (see Zacks [7], p. 247) the sampling distribution of these CMLE limits is approximately normal around the true prediction limits. Accordingly we may expect that in about 25–50% of the cases the CMLE prediction intervals (5.2) will be too small and will not cover the true prediction intervals. This is our main reason for developing alternative methods of estimation. On the other

hand, due to the large samples under consideration and the strong consistency of the maximum likelihood estimators (see Zacks [7]) we expect that the CMLE prediction limits will be close to the true limits, even if the intervals are not entirely covered. Numerical examples will be given to illustrate this point.

5.2 Tolerance Limits for the Prediction Intervals

Tolerance limits can be defined as confidence limits to fractiles of a distribution. We derive here tolerance limits for the prediction intervals (5.2) within a large sample framework. It is well known that in large binomial samples, if $\hat{\theta}$ is the CMLE of θ then $2\sin^{-1}\left(\sqrt{\hat{\theta}}\right)$ has an approximately normal distribution with expectation $2\sin^{-1}(\sqrt{\theta})$ and variance $1/n$, where n is the sample size (see Johnson and Kotz [3], p. 65). Since the conditional distribution of $X_{ijkl}(t+1)$ given $X_{ijkl}(t)$ is binomial with retention probability $\theta_{ijkl}(t)$ we consider the statistic

$$Y_{ijkl}(t)=2\sin^{-1}\left(\sqrt{\frac{X_{ijkl}(t)+.5}{X_{ijkl}(t-1)+1}}\right), \quad t\geq 2. \tag{5.3}$$

The mean of the asymptotic conditional normal distribution of $Y_{ijkl}(t)$, given $X_{ijkl}(t-1)$, is

$$H_{ijkl}(t)=2\sin^{-1}\left(\sqrt{\theta_{ijkl}(t)}\right), \tag{5.4}$$

and its asymptotic conditional variance is $1/X_{ijkl}(t-1)$.

Let $Y_{ijkl}(t+1)$ represent a statistic as in (5.3) based on $X_{ijkl}(t)$ and the yet unobserved $X_{ijkl}(t+1)$. The asymptotic conditional distribution of $Y_{ijkl}(t+1)$ given $X_{ijkl}(t)$ is, according to our assumption that $\theta_{ijkl}(t+1)=\theta_{ijkl}(t)$, normal with mean $H_{ijkl}(t)$ and variance $1/X_{ijkl}(t)$.

We determine now confidence limits for the unknow value of $H_{ijkl}(t)$. A $(1-\alpha/2)$ lower confidence limit for $H_{ijkl}(t)$ is, according to the normal approximation,

$$L_{ijkl}(t)=Y_{ijkl}(t)-z_{1-\alpha/2}/\sqrt{X_{ijkl}(t-1)}. \tag{5.5}$$

Similarly, a $(1-\alpha/2)$, upper confidence limit for $H_{ijkl}(t)$ is approximately

$$U_{ijkl}(t)=Y_{ijkl}+z_{1-\alpha/2}/\sqrt{X_{ijkl}(t-1)}. \tag{5.6}$$

We apply now the Bonferoni inequality

$$P[A\cap B]\geq 1-P(\bar{A})-P(\bar{B}), \tag{5.6}$$

where A is the event that the confidence limits (5.4) and (5.5) cover $H_{ijkl}(t)$ and B is the event that $Y_{ijkl}(t+1)$ belongs to the interval

$$H_{ijkl}(t)\pm z_{1-\alpha/2}/\sqrt{X_{ijkl}(t)}. \tag{5.7}$$

Accordingly, the probability is at least $1-2\alpha$ that the limits

$$Y_{ijkl}(t)\pm z_{1-\alpha/2}\left(\frac{1}{\sqrt{X_{ijkl}(t-1)}}+\frac{1}{\sqrt{X_{ijkl}(t)}}\right) \tag{5.8}$$

cover the future value $Y_{ijkl}(t+1)$. The limits will be called *tolerance limits*. Finally, since the function $2\sin^{-1}(\sqrt{\theta})$, for $0<\theta<1$, is strictly increasing we employ the inverse of this transformation on the two tolerance limits of $Y_{ijkl}(t+1)$ to obtain corresponding tolerance limits for $X_{ijkl}(t+1)$. These limits are:

$$\xi^{(U)}_{ijkl}(t)=(\sin(Y^{(U)}_{ijkl}/2))^2\cdot(X_{ijkl}(t)+1.)-.5, \tag{5.9}$$

and

$$\xi^{(L)}_{ijkl}(t)=(\sin(Y^{(L)}_{ijkl}/2.))^2(X_{ijkl}(t)+1.)-.5, \tag{5.10}$$

where $Y^{(U)}_{ijkl}(t)$ and $Y^{(L)}_{ijkl}(t)$ are the upper and lower tolerance limits of $Y_{ijkl}(t+1)$.

5.3 Bayes Prediction Limits

The forecasting of $X_{ijkl}(t+1)$, given the prior data, can be attempted also from a Bayesian point of view. The limits of the prediction intervals, as defined in Section 4, depend on the unknown retention probabilities $\theta_{ijkl}(t)$. In a Bayesian framework we ascribe these parameters prior distributions. Given the past observations we can derive the posterior distributions of $\theta_{ijkl}(t)$. Let $G^{(t)}_{ijkl}(\theta)$ denote the prior distribution function of $\theta_{ijkl}(t)$, $t=2, 3, \ldots$. We assume that for each (i, j, k, l) the retention probabilities $\theta_{ijkl}(t)$ and $\theta_{ijkl}(t+1)$ have the same prior distributions. Let $G^{(t)}_{ijkl}(\theta|X)$ denote the posterior distribution of $\theta_{ijkl}(t)$ given that $X_{ijkl}(t)=X$. According to this posterior distribution we can compute the posterior marginal distribution of $X_{ijkl}(t+1)$ given $X_{ijkl}(t)=X$. This distribution is called also the *predictive distribution* and is given by

$$\Pi^{(t)}_{ijkl}(\xi|X)=\int P[X_{ijkl}(t+1)\leq\xi|X_{ijkl}(t)=X, \theta_{ijkl}(t+1)=\theta]dG^{(t)}_{ijkl}(\theta|X). \tag{5.11}$$

Bayes prediction limits for $X_{ijkl}(t+1)$, at level $1-\alpha$, are values $\underline{\xi}$ and $\bar{\xi}$ such that

$$\Pi^{(t)}_{ijkl}(\bar{\xi}|X)-\Pi^{(t)}_{ijkl}(\underline{\xi}|X)\geq 1-\alpha. \tag{5.12}$$

Aitchison and Dunsmore [1] con ider for binomial distributions of future observations, beta-binomia predictive distributions, given the number of successes in the past (see [1], p. 24). These are predic tive distributions obtained by ascribing θ prior beta distributions. In our specific problem it woul be impractical to consider such beta-binomial predictive distributions from the numerical poin of view. Instead we apply the arcsin transformations (5.3). As before, the conditional (large sample distribution of $Y_{ijkl}(t+1)$ given $X_{ijkl}(t)$ is normal with expectation $H_{ijkl}(t)=2\sin^{-1}(\sqrt{\theta_{ijkl}(t)})$ and variance $1/X_{ijkl}(t)$. We assume that the prior distribution of $H_{ijkl}(t)$ is normal with expecta tion w and variance τ^2, i.e., $H_{ijkl}(t)\sim N(w, \tau^2)$. It follows then that the posterior distribution o $H_{ijkl}(t)$, given that $Y_{ijkl}(t)=y$ and $X_{ijkl}(t-1)=x$, is

$$H_{ijkl}(t)|Y_{ijkl}(t)=y, X_{ijkl}(t-1)=x\sim N\left(\frac{y\tau^2x+w}{1+\tau^2x}, \frac{\tau^2}{1+\tau^2x}\right). \tag{5.13}$$

The symbol $\sim$ designates "distributed like." (5.13) means that the posterior distribution of $H_{ijkl}(t$ given $Y_{ijkl}(t)=y$ and $X_{ijkl}(t-1)=x$ is normal, with mean $(y\tau^2x+w)/(1+\tau^2x)$ and varianc $\tau^2/(1+\tau^2x)$. We will consider the limiting posterior distribution obtained by letting $\tau^2\to\infty$ (diffused prior). Under this condition

$$H_{ijkl}(t)|Y_{ijkl}(t), X_{ijkl}(t-1)\sim N\left(Y_{ijkl}(t), \frac{1}{X_{ijkl}(t-1)}\right). \tag{5.14}$$

From (5.14) we obtain that the predictive distribution of $Y_{ijkl}(t+1)$, given $X_{ijkl}(t)$ and $X_{ijkl}(t-1)$ is

$$Y_{ijkl}(t+1)|X_{ijkl}(t), X_{ijkl}(t-1)\sim N\left(Y_{ijkl}(t), \frac{1}{X_{ijkl}(t)}+\frac{1}{X_{ijkl}(t-1)}\right). \tag{5.15}$$

Hence, the *Bayes prediction limits* for $Y_{ijkl}(t+1)$, given $X_{ijkl}(t)$ and $X_{ijkl}(t-1)$ are.

$$Y_{ijkl}(t)\pm z_{1-\alpha/2}\left\{\frac{1}{X_{ijkl}(t)}+\frac{1}{X_{ijkl}(t-1)}\right\}^{1/2}.$$

Notice that the interval given by (5.16) is always contained in the tolerance interval (5.8). If $w^{(U)}_{ijkl}$ and $w^{(L)}_{ijkl}(t)$ denote the upper and lower Bayes prediction limits in (5.16) then the Bayes prediction limits for $X_{ijkl}(t+1)$ are respectively

$$\zeta^{(U)}_{ijkl}(t)=(\sin\,(w^{(U)}_{ijkl}/2.))^2(X_{ijkl}(t)+1)-.5 \tag{5.17}$$

and

$$\zeta^{(L)}_{ijkl}(t)=(\sin\,(w^{(L)}_{ijkl}/2.))^2\,(X_{ijkl}(t)+1)-.5 \tag{5.18}$$

In Table 2 we provide numerical comparisons of the CMLE, tolerance and Bayes prediction limits for the cohort sizes in Phase II ($t>2$) on July 1, 1972. We provide also the actual size of the cohorts.

The m values corresponding to the various i values of Table 2 index the entry periods, in order of presentation in Table 1, excluding the July 71 entry period. Accordingly, $i=1$, $m=1$ corresponds to entry period $1-71_j$, $i=2$, $m=1$ corresponds to the entry period $1-70$, etc. As seen in Table 2, the three different systems of prediction limits yield (on a retrospective basis) close estimates of actual force size. The CMLE intervals are the smallest, and the tolerance intervals are the largest. As expected, the CMLE intervals might be too small and the actual coverage probability could, also, be too small. Indeed, in Table 2 the actual cohort sizes fall outside the CMLE intervals in 15 out of 36 cases, while they are not covered by the tolerance intervals in only four cases. When we sum the lower and the upper limits over all cohorts, we obtain more conservative prediction intervals for the total force in Phase II. We see in Table 2 that all the three types of prediction intervals (CMLE, tolerance and Bayes) yield total force intervals which cover the actual values.

6. ESTIMATING THE RETENTION RATE AFTER SIX MONTHS OF SERVICE

Cohorts which at the time of forecasting have been only six months in service should be considered separately, since generally the assumption $\theta_{ijkl}(t)=\theta_{ijkl}(t+1)$ is invalid for $t=1$ since these cohorts are just entering Phase II. In this case, however, we can estimate the retention probability $\theta_{ijkl}(2)$ on the basis of the retention rates of previous cohorts during the period $t=2$ (from six till twelve months). We propose the usage of the ratio estimator

$$\hat{\theta}_{ijkl}(2)=\frac{\sum_{v=1}^{l-1} X_{ijkv}(2)}{\sum_{v=1}^{l-1} X_{ijkv}(1)} \tag{6.1}$$

TABLE 2.—*Prediction limits for the size of cohorts on July 1, 1972* ($\alpha = .05$)

i	j	k	m	CMLE		Tolerance		Bayes		Actual
1	1	1	1	1401.	1434.	1382.	1446.	1393.	1439.	1418.
1	1	2	1	507.	520.	496.	523.	502.	521.	499.
1	2	1	1	3272.	3303.	3253.	3315.	3264.	3307.	3280.
1	2	2	1	676.	693.	665.	697.	671.	694.	674.
2	1	1	1	935.	979.	912.	997.	926.	986.	1009.
2	1	1	2	876.	916.	855.	932.	867.	922.	910.
2	1	1	3	1133.	1165.	1115.	1177.	1125.	1170.	1155.
2	1	2	1	113.	130.	103.	136.	109.	132.	135.
2	1	2	2	98.	111.	89.	114.	94.	112.	104.
2	1	2	3	158.	169.	149.	171.	154.	170.	163.
2	2	1	1	1973.	2009.	1952.	2023.	1964.	2014.	2019.
2	2	1	2	2277.	2309.	2257.	2322.	2269.	2314.	2301.
2	2	1	3	1112.	1133.	1099.	1140.	1106.	1136.	1121.
2	2	2	1	246.	260.	235.	264.	241.	261.	256.
2	2	2	2	223.	236.	214.	239.	219.	237.	230.
2	2	2	3	171.	181.	163.	183.	167.	182.	177.
3	1	1	1	1467.	1518.	1440.	1540.	1456.	1527.	1570.
3	1	1	2	1692.	1749.	1663.	1773.	1630.	1758.	1781.
3	1	1	3	2261.	2325.	2223.	2352.	2247.	2335.	2347.
3	1	1	4	2554.	2616.	2522.	2643.	2541.	2627.	2633.
3	1	1	5	2952.	3001.	2924.	3022.	2940.	3010.	2979.
3	1	2	1	147.	166.	136.	173.	142.	169.	165.
3	1	2	2	167.	187.	156.	194.	162.	190.	190.
3	1	2	3	266.	291.	252.	300.	260.	294.	292.
3	1	2	4	279.	300.	266.	307.	274.	302.	294.
3	1	2	5	391.	404.	380.	407.	386.	405.	386.
3	2	1	1	3333.	3379.	3307.	3399.	3322.	3387.	3403.
3	2	1	2	4623.	4676.	4594.	4699.	4611.	4685.	4675.
3	2	1	3	3380.	3427.	3354.	3447.	3369.	3435.	3416.
3	2	1	4	4115.	4159.	4090.	4177.	4104.	4166.	4108.
3	2	1	5	3207.	3240.	3187.	3252.	3199.	3245.	3207.
3	2	2	1	331.	348.	319.	353.	325.	350.	345.
3	2	2	2	431.	452.	417.	459.	425.	455.	451.
3	2	2	3	411.	432.	397.	440.	405.	435.	428.
3	2	2	4	466.	488.	453.	496.	460.	491.	478.
3	2	2	5	526.	540.	516.	544.	522.	541.	526.
Sums				48170	49249	47537	49656	47902	49404	49130

for every $l \geq 2$. Due to the independence of the random variables $X_{ijkl}(1)$ and $X_{ijkl}(2)$ as l varies, the conditional distribution of $\sum_{v=1}^{l-1} X_{ijkv}(2)$, given $\{X_{ijk1}(1), \ldots, X_{ijk,l-1}(1)\}$, is the binomial

$$B\left(\sum_{v=1}^{l-1} X_{ijkv}(1), \theta_{ijkl}(2)\right).$$

We assume here that $\theta_{ijkl}(2)$ is the same for all entry periods ($l=1, 2, \ldots$). Accordingly we can obtain CMLE, tolerance and Bayes prediction limits for $Y_{ijkl}(2)$, by employing the methods of Section 5 with the ratio estimator (6.1) rather than the estimator (5.1). The CMLE prediction limits are obtained from (5.2) by substituting (6.1) for $\hat{\theta}_{ijkl}(t)$. To obtain the tolerance limits for $X_{ijkl}(2)$ we define

$$Y^*_{ijkl}(2) = 2 \sin^{-1}\sqrt{\hat{\theta}_{ijkl}(2)}. \tag{6.2}$$

$(1-\alpha)$-confidence intervals for $H_{ijkl}(2)$ are then given, for large samples, by

$$Y^*_{ijkl}(2) \pm z_{1-\alpha/2} \Big/ \sqrt{\sum_{v=1}^{l-1} X_{ijkv}(1)}. \tag{6.3}$$

Tolerance limits for $Y_{ijkl}(2)$, which is the arcsin transform of $(X_{ijkl}(2)+.5)/(X_{ijkl}(1)+1.)$, are then

$$Y^*_{ijkl}(2) + z_{1-\alpha/2}\left\{\frac{1}{\sqrt{\sum_{v=1}^{l-1} X_{ijkv}(1)}} + \frac{1}{\sqrt{X_{ijkl}(2)}}\right\}. \tag{6.4}$$

Substituting in (5.9)—(5.10) the two limits obtained from (6.4) we obtain tolerance limits for $X_{ijkl}(2)$. In a similar manner we find that the Bayes prediction limits are

$$Y^*_{ijkl}(2) \pm z_{1-\alpha/2}\left\{\frac{1}{\sum_{v=1}^{l-1} X_{ijkv}(1)} + \frac{1}{X_{ijkl}(1)}\right\}^{1/2}. \tag{6.5}$$

Tolerance and Bayes prediction limits for $X_{ijkl}(2)$ are obtained by substituting the upper and lower limits of (6.4) and (6.5), respectively, in (5.9) and (5.10), or in (5.17) and (5.18). This approach is applied below to our data. In Table 3 we show the number of first-term enlistees (by cohort) that remained in service at epochs $t=1$ and $t=2$. We then apply the above formulae to forecast the number at $t=2$ on the basis of $t=1$. The resultant forecasts are shown in Table 4.

We see in Table 4 that the methods suggested in the present section provide prediction intervals which compare very well with the actual data. In most of the cases the values being predicted either lie within the prediction intervals or fall very close to them.

TABLE 3. *Number of enlistees remaining in service after 6 and 12 months*

Entry period l	Contract length	LHS—W		LHS—N		AHS—W		AHS—N	
(*1969*)		6m	12m	6m	12m	6m	12m	6m	12m
1	2 Yrs	6, 057	5, 631	1, 647	1, 544	12, 116	11, 648	2, 127	2, 017
2		3, 421	3, 146	1, 175	1, 080	10, 756	10, 385	1, 722	1, 637
1	3 Yrs	1, 767	1, 627	204	194	2, 374	2, 312	211	196
2		2, 376	2, 166	298	270	3, 603	3, 501	456	433
1	4 Yrs	3, 145	2, 853	441	408	4, 217	4, 081	471	456
2		3, 197	2, 864	371	334	5, 547	5, 371	586	554
(*1970*)									
3	2 Yrs	1, 440	1, 373	496	474	3, 915	3, 823	869	843
4		1, 562	1, 493	421	400	4, 269	4, 169	790	769
3	3 Yrs	1, 466	1, 372	195	187	2, 496	2, 228	302	292
4		1, 208	1, 143	134	131	2, 496	2, 437	257	251
3	4 Yrs	3, 370	3, 165	432	401	3, 934	3, 835	532	512
4		3, 333	3, 173	371	356	4, 527	4, 401	563	547
(*1971*)									
5	2 Yrs	1, 564	1, 489	537	525	3, 416	3, 351	720	702
6		1, 809	1, 717	744	716	4, 553	4, 451	1, 235	1, 195
5	3 Yrs	1, 293	1, 219	181	172	1, 180	1, 151	190	183
6		919	866	143	133	1, 034	997	233	227
5	4 Yrs	3, 325	3, 146	423	410	3, 366	3, 294	539	546
6		3, 359	3, 174	402	383	4, 445	4, 281	632	607
(*1972*)									
7	2 Yrs	2, 710	2, 585	1, 223	1, 177	3, 517	3, 413	1, 311	1, 273
8		1, 659	1, 576	729	688	3, 149	3, 075	1, 052	1, 014
7	3 Yrs	2, 067	1, 964	434	417	1, 329	1, 299	324	307
8		1, 766	1, 669	465	441	1, 568	1, 528	500	477
7	4 Yrs	4, 945	4, 696	940	893	3, 731	3, 620	684	655
8		4, 680	4, 407	1, 577	1, 514	4, 461	4, 305	1, 393	1, 328

TABLE 4. *Prediction limits for the size of cohorts at the beginning of Phase II.* ($\alpha = .05$)

i	j	k	l	CMLE		Tolerance		Bayes		Actual
1	1	1	2	3151.	3210.	3127.	3229.	3142.	3216.	3146.
1	1	1	3	1314.	1353.	1305.	1359.	1312.	1353.	1373.
1	1	1	4	1432.	1472.	1423.	1478.	1430.	1472.	1493.
1	1	1	5	1440.	1478.	1431.	1484.	1438.	1479.	1489.
1	1	1	6	1671.	1712.	1662.	1718.	1669.	1712.	1717.
1	1	1	7	2514.	2563.	2502.	2572.	2511.	2564.	2585
1	1	1	8	1539.	1577.	1532.	1581.	1537.	1577.	1576.
1	1	2	2	1085.	1118.	1069.	1129.	1079.	1121.	1080.
1	1	2	3	450.	472.	444.	475.	448.	472.	474.
1	1	2	4	383.	403.	378.	405.	382.	403.	400.
1	1	2	5	491.	514.	485.	516.	490.	513.	525.
1	1	2	6	687.	713.	680.	716.	685.	713.	716.
1	1	2	7	1139.	1170.	1129.	1176.	1136.	1171.	1177.
1	1	2	8	679.	703.	673.	705.	677.	702.	688.
1	2	1	2	10301.	10380.	10261.	10413.	10285.	10393.	10385.
1	2	1	3	3748.	3794.	3737.	3802.	3745.	3795.	3823.
1	2	1	4	4097.	4144.	4086.	4152.	4094.	4145.	4169.
1	2	1	5	3282.	3323.	3274.	3328.	3280.	3323.	3351.
1	2	1	6	4385.	4431.	4375.	4438.	4383.	4432.	4451.
1	2	1	7	3389.	3429.	3381.	3434.	3387.	3429.	3413.
1	2	1	8	3034.	3072.	3027.	3075.	3032.	3071.	3075.
1	2	2	2	1615.	1651.	1596.	1664.	1607.	1656.	1637.
1	2	2	3	812.	838.	804.	842.	810.	838.	843.
1	2	2	4	741.	765.	735.	768.	739.	765.	769.
1	2	2	5	678.	699.	672.	701.	676.	699.	702.
1	2	2	6	1170.	1197.	1162.	1201.	1167.	1197.	1195.
1	2	2	7	1244.	1272.	1237.	1276.	1242.	1272.	1273.
1	2	2	8	999.	1024.	993.	1026.	998.	1023.	1014.
2	1	1	2	2162.	2214.	2128.	2240.	2146.	2225.	2166.
2	1	1	3	1321.	1363.	1307.	1373.	1317.	1365.	1372.
2	1	1	4	1094.	1131.	1084.	1138.	1091.	1132.	1143.
2	1	1	5	1178.	1215.	1168.	1221.	1175.	1216.	1219.
2	1	1	6	838.	868.	831.	872.	836.	868.	866.
2	1	1	7	1899.	1944.	1886.	1953.	1895.	1946.	1964.
2	1	1	8	1628.	1669.	1618.	1675.	1625.	1670.	1669.
2	1	2	2	276.	291.	263.	295.	270.	293.	270.
2	1	2	3	173.	187.	166.	190.	171.	188.	187.
2	1	2	4	120.	131.	115.	131.	118.	130.	131.
2	1	2	5	164.	177.	159.	178.	162.	176.	172.
2	1	2	6	129.	140.	126.	141.	128.	140.	133.
2	1	2	7	399.	418.	391.	422.	396.	419.	417.
2	1	2	8	431.	450.	423.	453.	428.	450.	441.
2	2	1	2	3490.	3528.	3463.	3546.	3477.	3536.	3501.
2	2	1	3	2412.	2444.	2399.	2451.	2407.	2445.	2228.
2	2	1	4	2347.	2390.	2333.	2400.	2343.	2392.	2437.
2	2	1	5	1113.	1141.	1107.	1144.	1112.	1141.	1151.
2	2	1	6	977.	1002.	972.	1005.	976.	1002.	997.
2	2	1	7	1258.	1287.	1252.	1290.	1257.	1287.	1299.
2	2	1	8	1489.	1520.	1483.	1523.	1488.	1520.	1528.
2	2	2	2	413.	434.	392.	445.	402.	440.	433.
2	2	2	3	277.	293.	269.	296.	274.	293.	292.
2	2	2	4	237.	251.	232.	253.	236.	251.	251.
2	2	2	5	176.	187.	172.	188.	175.	187.	183.
2	2	2	6	217.	229.	213.	230.	215.	228.	227.
2	2	2	7	304.	318.	299.	319.	302.	317.	307.
2	2	2	8	470.	488.	463.	490.	468.	488.	477.

TABLE 4. *Prediction limits for the size of cohorts at the beginning of Phase II.* ($\alpha = .05$)—Continued

i	j	k	l	CMLE		Tolerance		Bayes		Actual
3	1	1	2	2868.	2932.	2832.	2962.	2853.	2944.	2864.
3	1	1	3	3004.	3072.	2977.	3094.	2995.	3079.	3165.
3	1	1	4	3017.	3080.	2996.	3096.	3010.	3084.	3173.
3	1	1	5	3043.	3103.	3026.	3116.	3038.	3105.	3146.
3	1	1	6	3090.	3148.	3075.	3160.	3086.	3150.	3174.
3	1	1	7	4571.	4640.	4551.	4656.	4566.	4644.	4696.
3	1	1	8	4343.	4409.	4327.	4422.	4339.	4411.	4407.
3	1	2	2	333.	353.	321.	359.	328.	355.	334.
3	1	2	3	383.	406.	373.	412.	379.	408.	401.
3	1	2	4	331.	351.	323.	355.	328.	352.	356.
3	1	2	5	382.	403.	375.	406.	380.	403.	410.
3	1	2	6	367.	386.	361.	389.	365.	386.	383.
3	1	2	7	869.	897.	858.	904.	865.	899.	893.
3	1	2	8	1468.	1504.	1453.	1514.	1463.	1507.	1514.
3	2	1	2	5342.	5394.	5309.	5419.	5327.	5405.	5371.
3	2	1	3	3787.	3830.	3771.	3841.	3781.	3833.	3835.
3	2	1	4	4369.	4414.	4354.	4424.	4364.	4416.	4401.
3	2	1	5	3248.	3286.	3237.	3292.	3245.	3287.	3294.
3	2	1	6	4298.	4341.	4286.	4349.	4295.	4342.	4281.
3	2	1	7	3600.	3641.	3591.	3647.	3597.	3641.	3620.
3	2	1	8	4306.	4351.	4296.	4358.	4304.	4351.	4305.
3	2	2	2	559.	576.	546.	581.	553.	578.	554.
3	2	2	3	499.	518.	490.	522.	496.	518.	512.
3	2	2	4	530.	549.	522.	552.	527.	549.	547.
3	2	2	5	509.	527.	503.	529.	507.	527.	546.
3	2	2	6	606.	622.	600.	624.	604.	622.	607.
3	2	2	7	654.	672.	649.	674.	652.	672.	655.
3	2	2	8	1335.	1361.	1325.	1366.	1332.	1362.	1328.

REFERENCES

[1] Aitchison, J. and I. R. Dunsmore, *Statistical Prediction Analysis* (Cambridge University Press, Cambridge, 1975).

[2] Haber, S. E., "Factors Influencing Attrition in the Marine Corps," Technical Paper Serial T–306, Program in Logistics, The George Washington University (1975).

[3] Johnson, N. L. and S. Kotz, *Distributions in Statistics: Discrete Distributions* (Houghton Mifflin Company, New York, 1969).

[4] Zacks, S. and S. E. Haber, "A Procedure for Forecasting the Size of a Force Subject to Random Withdrawals," Technical Paper Serial T–312, Program in Logistics, The George Washington University (1975).

[5] Zacks, S. and S. E. Haber, "Forecasting of Marine Corps Force Size as of December 31, 1974," Technical Memorandum Serial TM–80769, Program in Logistics, The George Washington University (1975).

[6] Zacks, S., "Forecasting the Total Size of the Marine Corps at the Beginning of April, July and October 1976," Technical Memorandum, Series TM–65705, Program in Logistics, The George Washington University (1976).

[7] Zacks, S., *The Theory of Statistical Inference* (John Wiley and Sons, New York, 1971).

A MODEL FOR THE DETERMINATION OF OPTIMAL INTERFIRING TIMES

Yoram Friedman

Tel-Aviv University
Tel-Aviv, Israel

ABSTRACT

Various models of stochastic duels with round dependent hit probabilities have appeared in the literature [1]. However, none of them analyzed the effect interfiring times will have on the hit probability.

In this paper we formulate a model for a marksman versus a passive target where the probability of a hit at a given round is a function of the interfiring times.

We show how to solve for the optimal interfiring times and prove that under certain assumptions the optimal rate of fire is a non-decreasing function of the round fired.

I. INTRODUCTION

Various models of stochastic duels with round dependent hit probabilities have appeared in the literature. (For a detailed discussion and references see [1].) However, none of them analyzed the effect interfiring times will have on the hit probability.

Very often, especially in an armored combat, one of the important parameters which determine the result of the duel is the rate of fire, and it is common practice to advocate the highest possible rate of fire.

However, this policy will not always be optimal, especially when the firing unit needs some minimal time to make corrections. Thus the gain achieved from a high rate of fire (short interfiring time) might be overwhelmed by the loss in accuracy.

In this paper we formulate a model for a marksman firing at a passive target where the probability of a hit in a given round is a function of the interfiring times, a parameter α, and the first round failure probability. The total firing time is also limited and the interfiring times are fixed, not stochastic.

We show that under the assumption of a constant rate of learning, the optimal policy is to increase the rate of fire with every round fired.

II. THE MODEL

Let $P(i)$ be the probability of hitting the target on the i^{th} round, given all the previous rounds have missed. Let $q(i)=1-P(i)$. We assume the following learning process:

$$q(i)=(\alpha+(1-\alpha)e^{-t_i})q(i-1) \qquad i=2, 3, \ldots \tag{1}$$

where t_i is the interfiring time between the $(i-1)^{\text{th}}$ round and the i^{th} round and $0\leq\alpha\leq 1$. (Notice that $q(i)$ depends on the first round failure probability by $q(i)=q(1)\prod_{j=2}^{i}(\alpha+(1-\alpha)e^{-t_i})$).

Thus, the probability of a miss at the i^{th} round is non-increasing in $i(q(i)\leq q(i-1))$ and depends on the magnitude of α and t_i.

If t_i is zero then no learning takes place (since there is no time for corrections) and $q(i)=q(i-1)$. As t_i gets bigger, the influence of learning increases, but at a diminishing rate. For $t_i=\infty$, $q(i)=\alpha q(i-1)$, which means that α can be interpreted as the maximal relative improvement that can be achieved in a given round.

We assume that the firing unit cannot hold the same position indefinitely (a common practice in tank duels). This means that if the target is not hit in T units of time, the firing unit has to change its position and restart the firing process.

We also assume that there is a minimal time, t, to load and fire the next round, and that during that time learning takes place; thus we have $t_i\geq t$.

Notice also that the probability of hit at the first round is an external parameter to the model and that the maximal time in position (T) is counted only after the first shot has been fired. This means that we assume a tactical situation such that the firing unit is not discovered by the enemy before firing the first shot. These assumptions about T and t give an upper bound to the maximum number of rounds that can be fired from the same position. Denoting this number by M we have that

$$M=1+\left[\frac{T}{t}\right]$$

where $[T/t]$ means the greatest integer less than or equal to T/t. The objective of the firing unit is to maximize the probability of hitting the target before a change in position will have to take place.

III. SOLUTION PROCEDURE

Let n be the number of rounds the unit fires from the same position, $1\leq n\leq M$. For any given n, we have to find the optimal interfiring times $t_2 \ldots, t_n$.

The probability of hitting the target in less than n rounds is:

$$P_H(n)=1-\prod_{i=1}^{n}q(i) \tag{2}$$

and the optimization problem for a given n is then:

$$\min \prod_{i=1}^{n}q(i)$$

$$\text{s.t.} \sum_{i=2}^{n}t_i=T \tag{3}$$

$$t_i\geq t \qquad i=2, \ldots, n.$$

We treat the first constraint as an equality since it is never optimal not to use all the available time.

Taking the logarithm of the objective function in (3) we get the equivalent problem:

$$\min \sum_{i=1}^{n} \ln q(i)$$

$$\text{s.t.} \sum_{i=2}^{n} t_i = T \tag{4}$$

$$t_i \geq t \qquad i=2, \ldots, n.$$

Using (1), we have that the objective function in (4) can be written as:

$$\sum_{i=1}^{n} \ln q(i) = n \ln q(1) + \sum_{i=2}^{n} (n-i+1) \ln(\alpha + (1-\alpha)e^{-t_i}) \tag{5}$$

The objective function in (4) is strictly convex and thus the necessary and sufficient conditions for optimality are:

$$\frac{(n-i+1)(1-\alpha)e^{-t_i}}{\alpha+(1-\alpha)e^{-t_i}} = \lambda \qquad \text{if } t_i > t \tag{6}$$

$$\frac{(n-i+1)(1-\alpha)e^{-t_i}}{\alpha+(1-\alpha)e^{-t_i}} \leq \lambda \qquad \text{if } t_i = t \tag{7}$$

$$\sum_{i=2}^{n} t_i = T \tag{8}$$

$$t_i \geq t \qquad i=2, \ldots, n \tag{9}$$

where $\lambda > 0$ is the Lagrange multiplier associated with constraint (8).

The following theorem establishes the structure of the optimal solution:

THEOREM: The optimal solution to problem (4) is such that

$$t_2 \geq t_3 \geq \ldots \geq t_n \geq t.$$

PROOF: Let $g(x) = \dfrac{(1-\alpha)e^{-x}}{\alpha+(1-\alpha)e^{-x}}$

If we have for some i and j $(j>i)$ that $(n-i+1)g(t_i)=\lambda$ and $(n-j+1)g(t_j)=\lambda$ then, since $g(t_i)$ is non-increasing in t_i, we must have $t_i \geq t_j$. Also if for some k, $(n-k+1)g(t) \leq \lambda$ then for any $l>k$ $(n-l+1)g(t) \leq \lambda$.

Q.E.D.

The theorem above enables us to establish the following procedure for finding the optimal solution:

1. Let $k=n$
2. Let $t_i > t$ for $i=2, \ldots, k$. From (6) we get that

$$t_i = \ln \frac{1-\alpha}{\alpha} + \ln\left(\frac{n-i+1}{\lambda} - 1\right) \qquad i=2, \ldots, k \tag{10}$$

3. Solve for λ using equation (8) and (10) i.e.

$$\sum_{i=2}^{n} t_i=(n-1)\ln\frac{1-\alpha}{\alpha}+\sum_{i=2}^{n}\ln\left(\frac{n-i+1}{\lambda}-1\right)=T$$

If there exists $\lambda>0$ that solves the above equation, we have an optimal solution. Otherwise go to 4.

4. Set $t_k=t$. $k=k-1$, $T=T-t$ and go to 2.

Recall that up to now we have solved the problem of allocating the interfiring times for a given n. To find the optimal solution for the overall problem we solve the sub-problem (where n is given) for all values of n, $2\leq n\leq M$ and choose the one that gives the best value of the objective function. (Usually M is rather small, so that the number or problems to be solved is of no practical concern).

IV. NUMERICAL EXAMPLE

As an example of the above algorithm, consider the following situation:

A tank is ambushing an approaching enemy force. According to previous experience the tank can remain safely in the same position for 30 seconds after firing the first round. The minimum time required for reloading and correcting the previous shot is 10 seconds.

Assume the first round failure probability is $q(1)=.80$ and that the maximal relative improvement that can be achieved in a given round is $\alpha=.50$.

Taking our basic unit of time to be 10 seconds, we have $t=1$, $T=3$ and $M=4$.

Solving the problem for value of $n=2, 3, 4$, we get the following results:

n	t_2	t_3	t_4	$P_H(n)$
2	3.00	----------		.66
3	1.93	1.07	--------	.89
4	1.00	1.00	1.00	.96

Thus we see that for the above example the optimal strategy is to fire 4 rounds with the minimal interfiring times between rounds.

V. SUMMARY

A model for the optimal allocation of interfiring times was proposed. The model takes into account the learning effect that is present in a sequence of shots to the same target and thus is particularly suitable for an armored combat type of battle.

It was shown that under the assumption of a constant learning coefficient, the rate of fire should increase with the number of rounds fired. The model enables the tactician to specify the time that should be invested in corrections and the maximum number of rounds that should be fired from the same position.

Also, by solving the problem for different values of T, one can determine the desired maximal time a unit should stay in the same position.

REFERENCES

[1] Ancker, C. J. Jr., "Stochastic Duels with Round Dependent Hit Probabilities," Naval Research Logistics Quarterly, *22*, 575–583 (1975).

A NOTE ON THE RELATIONSHIP BETWEEN CORRELATION AND CIRCULAR ERROR PROBABILITY (CEP)

Dale E. Olsen

Applied Physics Laboratory
The Johns Hopkins University
Laurel, Maryland

ABSTRACT

This note investigates the effects of ignoring correlation (ρ) as is sometimes done when estimating or placing confidence intervals on circular error probability (CEP). It is shown that better estimates of CEP can be made if the axis is rotated [5] so that the estimate of ρ is zero. It is also shown that ignoring variability in estimating correlation can affect the confidence intervals on CEP or the placing of a lower bound on the probability content of a circle.

1. INTRODUCTION

The accuracy of a weapon system is often described using the radius of a circle containing 50 percent of the hitting probability and centered at the target. The radius is called CEP (or CPE by Blischke and Halpin [2]). Suppose a Cartesian coordinate system is located at the target and let (X, Y) denote the coordinates of an impact of a projectile. It is usually assumed that the hitting probability follows a bivariate normal distribution with parameters μ_x, μ_y, σ_x^2, σ_y^2, and ρ. In many cases the correlation between X and Y will be non-zero. For example, if one of the principal contributors to system miss affects both the X and Y components, a large positive value of the contributor may result in large positive values of X and Y, while a large negative value of the contributor may result in large negative values of X and Y. This would result in the correlation being positive.

This note shows that correlation should not be ignored, as is sometimes done when dealing with CEP, by illustrating the relationship between CEP and correlation. If CEP is calculated based on modeled (e.g., from design specifications) values of the bivariate normal parameters and a non-zero correlation is assumed to be zero, then the calculated value of CEP will be incorrect. Similarly, if μ_x, μ_y, σ_x^2, σ_y^2 are estimated using impact data and ρ is assumed to be zero, the estimator for CEP may be poor. In Section 2 the effects of ignoring correlation in calculating CEP will be illustrated. The uncertainty in estimating correlation should also be considered (1) when placing a confidence interval on CEP or (2) when placing a lower bound on the probability contained in a circle of fixed radius. The probabilities associated with either the confidence interval or with the

circle can be significantly affected by sampling errors in estimating correlation as is demonstrated in Section 3.

Guenther and Terragno [5] have reviewed and summarized the literature on CEP prior to their paper in 1964. More recently, Blischke and Halpin [2] have suggested methods for placing confidence intervals on CEP. Zacks and Solomon [8] have suggested a method for placing lower confidence limits on impact probability in a circle of fixed radius when $\mu_x=\mu_y=0$, and $\sigma_x^2 \leq \sigma_y^2$. Their procedure can be used to place confidence intervals on CEP by searching iteratively for the radius of a circle with impact probability of 0.50.

2. ESTIMATION

An estimate of CEP can be found by first estimating the parameters of a bivariate normal distribution, so that a density function can be estimated. The density function is then integrated [3, 4] over the circles with centers at the target until one is found which contains 50 percent of the probability. A second method is to use tables of offset circle probabilities such as those provided by Biser [1]. Both the integration and the tables techniques require a change of variables (or a rotation of the axis) [6] so that the estimated correlation is zero. This step is sometimes skipped so that approximating formulas can easily be used. Skipping this step can result in poor estimates of CEP.

Figures 1 through 7 and Table 1 illustrate the relationship between correlation and CEP. These charts were generated by first changing the variables of integration so that correlation was zero and then integrating the new density function over circles. They illustrate the effect of ignoring correlations when estimating CEP. For example, if our best estimates of the parameters

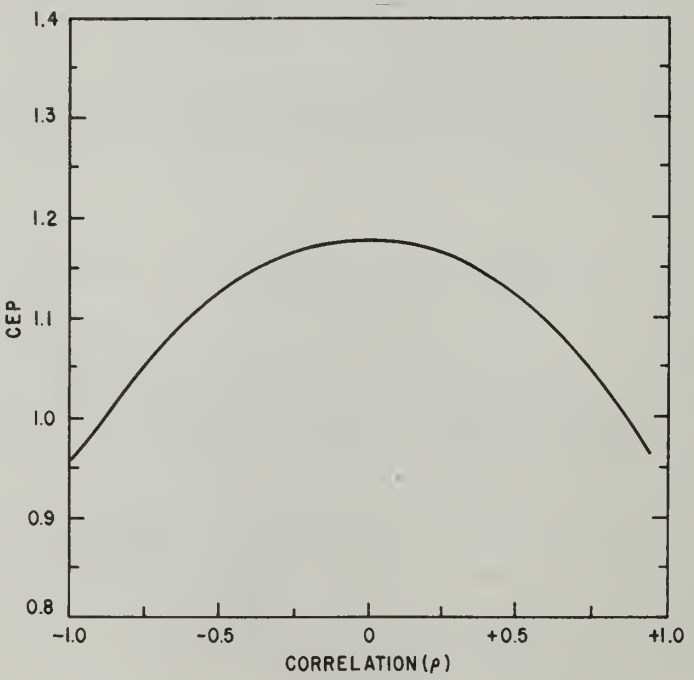


FIGURE 1. Plot of CEP$=f(\rho)$

$\mu_x=\mu_y=0$ $\sigma_x=\sigma_y=1$

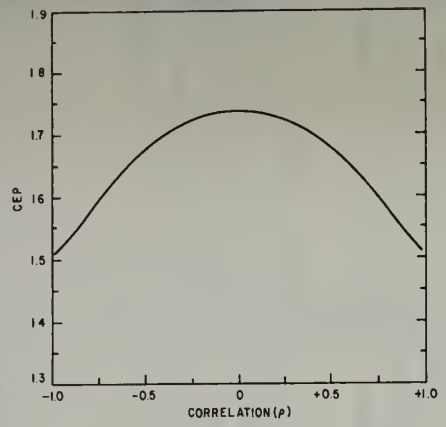


FIGURE 2. Plot of CEP=$f(\rho)$

$\mu_x=\mu_y=0$ $\sigma_x=1$, $\sigma_y=2$

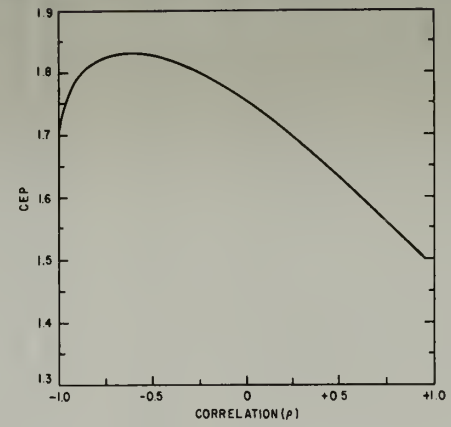


FIGURE 4. Plot of CEP=$f(\rho)$

$\mu_x=\mu_y=\sigma_x=\sigma_y=1$

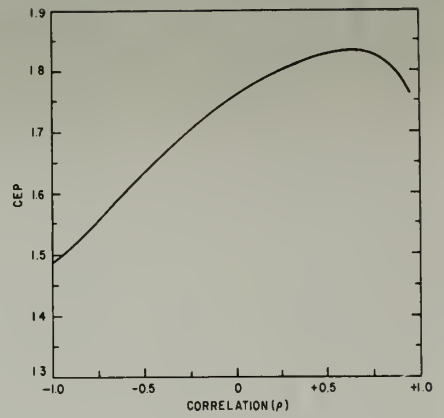


FIGURE 6. Plot of CEP=$f(\rho)$

$\mu_x=1$, $\mu_y=-1$ $\sigma_x=\sigma_y=1$

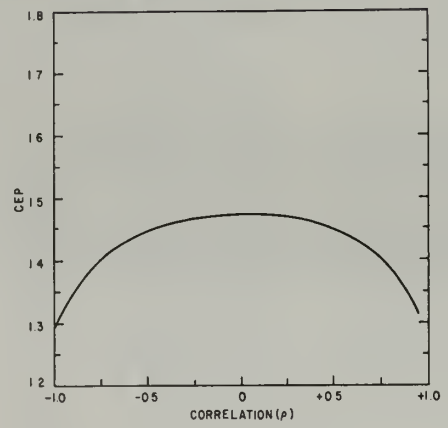


FIGURE 3. Plot of CEP=$f(\rho)$

$\mu_x=0$ $\mu_y=1$ $\sigma_x=\sigma_y=1$

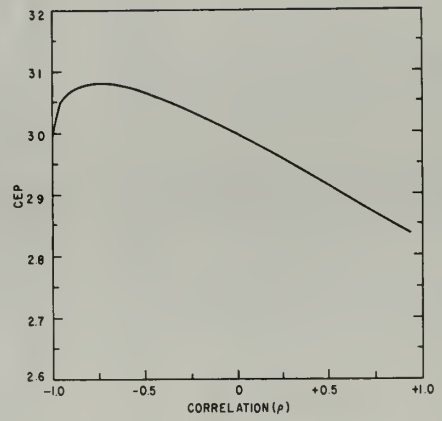


FIGURE 5. Plot of CEP=$f(\rho)$

$\mu_x=\mu_y=2$ $\sigma_x=\sigma_y=1$

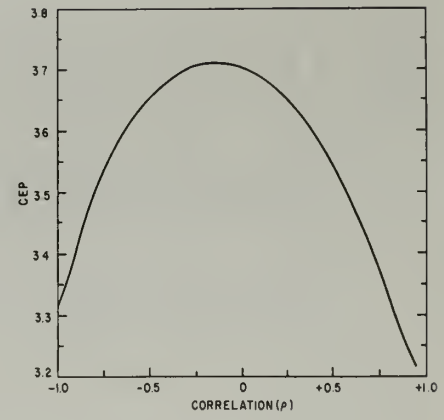


FIGURE 7. Plot of CEP=$f(\rho)$

$\mu_x=\mu_y=-1$, $\sigma_x=2$ $\sigma_y=4$

TABLE 1. *The Effect of correlation on CEP*

ρ	CEP	CEP assuming $\rho=0.0$	Percent over estimated	μ_x	μ_y	σ_x	σ_y
+1.00	0.954	1.177	23.4	0	0	1	1
+0.80	1.031	1.177	14.2	0	0	1	1
+0.60	1.100	1.177	7.0	0	0	1	1
+0.40	1.145	1.177	2.8	0	0	1	1
+0.20	1.170	1.177	0.6	0	0	1	1
−0.20	2.242	2.247	0.2	0	2	1	1
−0.40	2.227	2.247	0.9	0	2	1	1
−0.60	2.199	2.247	2.2	0	2	1	1
−0.80	2.147	2.247	4.7	0	2	1	1
−1.00	2.052	2.247	9.5	0	2	1	1
−1.00	1.705	1.759	3.2	1	1	1	1
−0.75	1.826	1.759	−3.7	1	1	1	1
−0.50	1.828	1.759	−3.8	1	1	1	1
−0.25	1.802	1.759	−2.4	1	1	1	1
+0.25	1.702	1.759	3.3	1	1	1	1
+0.50	1.632	1.759	7.8	1	1	1	1
+0.75	1.554	1.759	13.2	1	1	1	1
+1.00	1.486	1.759	18.4	1	1	1	1
−1.00	2.241	2.457	9.6	1	−2	1	1
−0.75	2.301	2.457	6.8	1	−2	1	1
−0.50	2.362	2.457	4.0	1	−2	1	1
−0.25	2.415	2.457	1.7	1	−2	1	1
+0.25	2.488	2.457	−1.2	1	−2	1	1
+0.50	2.597	2.457	−2.0	1	−2	1	1
+0.75	2.502	2.457	−1.6	1	−2	1	1
+1.00	2.379	2.457	3.3	1	−2	1	1

are $\hat{\mu}_x=\hat{\mu}_y=0$, $\hat{\sigma}_x=\hat{\sigma}_y=1$ and $\hat{\rho}=0.8$ (see Table 1), our best estimate of CEP is 1.031. However, if we assume that $\rho=0$, we get the inflated estimate of CEP=1.177. This last estimate is over 14 percent too large. Table 1 gives additional examples of the effects of ignoring correlation. Column 2 of this table gives the value of CEP for the parameter values given in columns 1, 5, 6, 7, 8. Column 3, labeled CEP, assuming $\rho=0$, gives the value of CEP if the correlation coefficient were zero and the other parameters had the same values. Column 4 gives the percentage that the CEP would be overestimated had the actual value of ρ been given in column 1 and had CEP been computed assuming $\rho=0$.

Figures 1 and 2 show that when $\mu_x=\mu_y=0$, the largest value of CEP is obtained when $\rho=0$. This remains true as long as at least one mean (μ_x or μ_y) is zero. The curves are not symmetrical unless $\sigma_x=\sigma_y$ and one mean is zero. If both means are non-zero, the maximum value of CEP is not at $\rho=0$ as shown in Figures 4 through 7. The value of ρ giving the maximum values of CEP is positive when μ_x and μ_y have different signs and is negative when μ_x and μ_y have the same sign.

3. CONFIDENCE INTERVALS

Blischke and Halpin [2] describe a method for placing a lower bound on CEP. Their method does not assume $\rho=0$. Zacks and Solomon [8] describe a method of placing a lower bound on the

probability contained in a circle of a fixed radius. Their procedure assumes $\rho=0$. If the correlation is not known or not assumed to be zero, they suggest rotating the axis (using the eigenvalues of the covariance matrix) so that the estimate of ρ is zero. Their procedure can be used for placing a confidence bound on CEP by iteratively searching for a circle that contains at least 50 percent of the hitting probability at some given confidence level. Although their procedure gives good results when correlation is zero, it does not account for the variability in correlation and therefore can give misleading results.

An example will now be given which illustrates how assuming that the estimated correlation is correct can lead to an incorrect lower bound on hitting probability. Suppose ten shots were taken at a target and the parameter estimates are $\hat{\rho}=0.55$, $\hat{\mu}_x=\hat{\mu}_y=0$, $\hat{\sigma}_x=\hat{\sigma}_y=1$. A one-sided test of $\rho=0.0$ would not be rejected at the 5-percent level. The Zacks and Solomon 90 percent *lower coverage probability* for a circle of radius 1.0 would be 0.327. However, if the true correlation were zero, as could reasonably be the case, the Zacks and Solomon 90 percent lower coverage probability for the same circle should be 0.291. Since assuming $\hat{\rho}=\rho$ can lead to errors in computing lower coverage probabilities, the variability in $\hat{\rho}$ should be considered when developing procedures for placing a lower bound on a coverage probability.

Table 2 contains additional examples of the variability in the lower coverage probabilities that can be caused by the variability in the estimate of correlation. In this table the Zacks and Solomon technique was used to find the 90 percent lower coverage probabilities for a circle of radius one when $\hat{\mu}_x=\hat{\mu}_y=0$ and $\hat{\sigma}^2_x=\hat{\sigma}^2_y=1$ and for various sample sizes and estimated correlations. For estimated correlations ($\hat{\rho}$) of 0.0, 0.20, 0.40, and 0.60, the corresponding 95 percent

TABLE 2. *Zacks and Solomon lower 90% coverage probabilities* for a circle of radius one for three estimates of correlation***

Sample size	Coverage	$\underline{\rho}$	Coverage	$\hat{\rho}$	Coverage	$\bar{\rho}$
10	0. 1417	−0. 60	0. 1172	0. 00	0. 1417	0. 60
10	0. 1306	−0. 47	0. 1193	0. 20	0. 1569	0. 71
10	0. 1221	−0. 30	0. 1264	0. 40	0. 1766	0. 80
10	0. 1173	−0. 05	0. 1417	0. 60	0. 2051	0. 88
15	0. 2095	−0. 50	0. 1873	0. 00	0. 2095	0. 50
15	0. 1967	−0. 34	0. 1904	0. 20	0. 2268	0. 63
15	0. 1888	−0. 14	0. 2007	0. 40	0. 2494	0. 74
15	0. 1884	−0. 12	0. 2221	0. 60	0. 2811	0. 84
20	0. 3254	−0. 43	0. 3045	0. 00	0. 3254	0. 43
20	0. 3117	−0. 26	0. 3087	0. 20	0. 3458	0. 58
20	0. 3047	−0. 05	0. 3224	0. 40	0. 3724	0. 71
20	0. 3091	0. 21	0. 3493	0. 60	0. 4035	0. 82
50	0. 3457	−0. 27	0. 3375	0. 00	0. 3457	0. 27
50	0. 3382	−0. 08	0. 3419	0. 20	0. 3616	0. 45
50	0. 3415	0. 14	0. 3562	0. 40	0. 3837	0. 60
50	0. 3562	0. 40	0. 3836	0. 60	0. 4166	0. 75

*It is assumed that $\hat{\mu}_x=\hat{\mu}_y=0$ and $\hat{\sigma}_x{}^2=\hat{\sigma}_y{}^2=1$.

**$\hat{\rho}$ is the usual estimate of correlation $\underline{\rho}$ and $\bar{\rho}$ are the lower and upper 95% interval estimates of correlation btained from (7).

lower and upper confidence limits on the correlation coefficient ($\underline{\rho}$, $\overline{\rho}$) were found. These limits represent a reasonable extreme in the true value of correlation. For each of the values of correlation ($\underline{\rho}$, $\hat{\rho}$, $\overline{\rho}$), the 90 percent lower coverage probability was tabulated. To clarify the ideas, consider an example. For a sample size of 10 and $\hat{\rho}=0.6$, the 95 percent confidence interval on ρ is $(-0.05, 0.88)$. That is, the true correlation could reasonably be as small as -0.05 or as large as 0.88. The lower coverage probability obtained by assuming $\rho=-0.05$ is 0.1173 and by assuming $\rho=0.88$ is 0.2051. These numbers illustrate the large amount of variability in coverage probabilities possible because of the uncertainty in correlations.

4. CONCLUSION

Correlation is sometimes assumed to be zero when calculating CEP. This assumption may be made to facilitate use of a formula for computing CEP. If the assumption is reasonable, the resulting CEP will be reasonably accurate. However, if correlation is high, the errors in calculating CEP may be considerable. Since CEP does depend on correlation, as well as on the other four bivariate normal parameters, the uncertainty in correlation should be considered in the development of confidence limits for CEP and in placing a lower bound on the probability contained in a circle of fixed radius.

5. ACKNOWLEDGMENT

I would like to thank Herbert Solomon for his many helpful suggestions in preparing this paper.

REFERENCES

[1] Biser, Erwin, "Tables of Offset Circle Probabilities for a Normal Bivariate Elliptical Distribution," U.S. Army Electronics Command, Fort Monmouth, New Jersey, AD 623 882 (1965).

[2] Blischke, W. R., and A. H. Halpin, "Asymptotic Properties of Some Estimators of Quantiles of Circular Error," Journal of the American Statistical Association, *61*, 618–632 (1969).

[3] Di Donato, A. R., and M. P. Jarragin, "Integration of the General Bivariate Gaussian Distribution Over an Offset Circle," Mathematics of Computation, *15*, 375–382 (1964).

[4] Gilliland, D. C., and E. R. Hansen, "A Note on Some Series Representation of the Integral of a Bivariate Normal Distribution Over an Offset Circle," Naval Research Logistics Quarterly, *21* (1974).

[5] Guenther, W. C., and P. G. Terragno, "A Review of the Literature on a Class of Coverage Problems," Annals of Mathematical Statistics, *35*, 232–260 (1964).

[6] Hald, A., *Statistical Theory With Engineering Applications*, (John Wiley and Sons, Inc., New York, p. 596).

[7] Pearson, E. S. and H. O. Hartley, *Biometrika Tables for Statisticians, I*, (Cambridge University Press, New York 1970, p. 148).

[8] Zacks, S., and H. Solomon, "Lower Confidence Limits for the Impact Probability Within a Circle in the Normal Case," Naval Research Logistics Quarterly, *22* (1975).

A NOTE ON VALUES AND MULTILINEAR EXTENSIONS

Alvin E. Roth

University of Illinois
Urbana, Illinois

ABSTRACT

An axiomatic formulation is given of a class of values for cooperative games. This class includes the Shapley value and the Banzhaf index, and is related to the multilinear extension of a game.

In a recent paper [8], Owen extends the definition of the Banzhaf* index [1, 2] to non-simple games. He characterises it in terms of the multilinear extension of games, which permits an intriguing comparison with the characterization of the Shapley value which he obtained in [7].

Here we will develop an axiomatic formulation if this index which will permit comparisons to be made with the original formulation of the Shapley value [12], with the recent interpretations of the Shapley value as a cardinal utility [11], and with recent treatments of the Banzhaf and related indices for simple games [3, 5, 6, 10]. We will also characterize a wide class of related values.

For a set of players $N=\{1, \ldots, n\}$, a game in characteristic function form is a real valued function on the subsets of N, such that $v(\Phi)=0$. For a given game v, a coalition $T\subset N$ is a carrier if for all $S\subset N$, $v(S)=v(T\cap S)$. If π is a permutation of N, then for all $S\subset N$ we denote the image of S under π by πS, and define the game πv by $\pi v(\pi S)=v(S)$.

For each player $i\epsilon N$ the *extended* † *Banzhaf-Coleman index* for a game v is defined by

$$\psi_i(v)=\sum_{\substack{S\subset N\\ i\epsilon S}}\frac{1}{2^{n-1}}\left[v(S)-v(S-i)\right] \tag{1}$$

Thus for each player i in a game v, the extended Banzhaf-Coleman index is a weighted sum of the contributions which player i makes to each coalition of which he is a member.

The sum over all players of such contributions will be denoted by

$$T(v)=\sum_{i\epsilon N}\sum_{\substack{S\\ i\epsilon S}}[v(S)-v(S-i)]$$

*The definitions given by Coleman in [4] are also closely related.

†We use the term "extended" to denote the fact that the original definitions of Banzhaf and Coleman applied only to simple games, while equation (1) extends the definition to non-simple games.

We will show that $\psi=(\psi_1, \ldots \psi_n)$ is the unique function $f=(f_1, \ldots f_n)$ which satisfies the following conditions. (Note the similarity of these conditions both to the axioms which define the Shapley value [12], and to the axioms which determine the Banzhaf-Coleman index for *simple* games)[5].

CONDITION 1: (Symmetry) For each permutation π, $f_{\pi i}(\pi v)=f_i(v)$.

CONDITION 2: (Additivity) For any games v and w, $f(v+w)=f(v)+f(w)$.

CONDITION 3: For each carrier S of v, $\sum_{i\in S} f_i(v)=\frac{T(v)}{2^{n-1}}$

THEOREM 1: The extended Banzhaf-Coleman index is the unique function which satisfies Conditions 1, 2, and 3.

For each $R\subset N$, we will find it convenient to consider games of the form cv_R defined by $cv_R(S)=\begin{cases} c \text{ if } R\subset S \\ 0 \text{ otherwise.}\end{cases}$ In general we will denote the cardinality of sets R, S, T by r, s, t.

LEMMA 1: If f obeys conditions 1 and 3, then

$$f_i(cv_R)=\begin{cases} c/2^{r-1} & \text{if } i\in R \\ 0 & \text{if } i\notin R\end{cases}$$

PROOF: Suppose $i\notin R$. Then R and $R\cup i$ are both carriers of the game v_R, so Condition 3 implies $f_i(cv_R)=0$. But if $i, j\in R$, then Condition 1 implies $f_i(cv_R)=f_j(cv_R)$. So for $i\in R$,

$$f_i(cv_R)=\frac{1}{r}\frac{T(cv_R)}{2^{n-1}}=\frac{c2^{n-r}}{2^{n-1}}=\frac{c}{2^{r-1}}.$$

PROOF OF THEOREM: Every game v is a sum of games of the form cv_R. In fact (cf.[12]), we can write $v=\sum_{R\subset N} c_R v_R$ where

$$c_R=\sum_{T\subset R}(-1)^{r-t}v(T). \tag{2}$$

Thus if f is a function which obeys Conditions 1, 2, and 3, then Lemma 1 and Condition 2 imply that f is given by

$$f_i(v)=\sum_{R\subset N} f_i(c_R v_R)=\sum_{\substack{R\subset N \\ i\in R}} c_R\frac{1}{2^{r-1}}.$$

So by equation (2), we have

$$f_i(v)=\sum_{\substack{R\subset N \\ i\in R}}\sum_{T\subset R}(-1)^{r-t}v(T)\frac{1}{2^{r-1}}=\sum_{T\subset N}\left(\sum_{\substack{R\subset N \\ R\supset\{T\cup i\}}}(-1)^{r-t}\frac{1}{2^{r-1}}\right)v(T).$$

If we denote the term in parentheses by $g_i(T)$, then

$$f_i(v)=\sum_{\substack{T\subset N \\ i\in T}} g_i(T)[v(T)-v(T-i)]. \tag{3}$$

But, since $i\in T$, there are $\binom{n-t}{r-t}$ coalitions R of size r such that $R\supset\{T\cup i\}$, so

$$g_i(T)=\sum_{r=t}^{n}(-1)^{r-t}\binom{n-t}{r-t}\frac{1}{2^{r-1}}.$$

Letting $q=r-t$ and $m=n-t$ we have

$$g_i(T)=\sum_{q=0}^{m}(-1)^q\binom{m}{q}\frac{1}{2^{q+t-1}}.$$

So by the binomial theorem,

$$g_i(T)=\frac{1}{2^{t-1}}\left(1-\frac{1}{2}\right)^{n-t}=\frac{1}{2^{n-1}}.$$

Substituting this into equation (3), we see that it is the same as equation (1). Thus equation (1) gives the only function which might obey Conditions 1, 2, and 3. It is easy to confirm that ψ obeys the conditions, which completes the proof.

It should be noted that the conditions which determine the extended Banzhaf-Coleman index are very similar to those which define the Shapley value. If the right-hand side of Condition 3 were changed from $T(v)/2^{n-1}$ to $v(T)$, then the modified conditions would define the Shapley value, given by the familiar formula

$$\phi_i(v)=\sum_{S\subset N}\frac{(s-1)!(n-s)!}{n!}[v(S)-v(S-i)].$$

Thus both the Shapley value and the extended Banzhaf-Coleman index obey Conditions 1 and 2, and both are weighted sums of the terms $[v(S)-v(S-i)]$. Both functions also obey the following, weaker version of Condition 3.

CONDITION 3a: For any carriers T, S, of v, $\sum_{i\in T}f_i(v)=\sum_{i\in S}f_i(v)$.

In general, we can state the following theorem.

THEOREM 2: A function $f=(f_1, \ldots, f_n)$ obeys Conditions 1, 2, and 3a if and only if

$$f_i(v)=\sum_{\substack{T\subset N\\ i\in T}}k_i(t)[v(T)-v(T-i)] \tag{4}$$

where k_t is a real valued function on $t=1, \ldots, n$.

PROOF: Clearly a function f given by equation (4) satisfies Conditions 1 and 2. It also satisfies Condition 3a, since if S is a carrier for v and $j\notin S$, then $f_j(v)=0$. So if $\hat{S}$ is the minimum carrier of v, then $\sum_{i\in S}f_i(v)=\sum_{i\in\hat{S}}f_i(v)$.

If f obeys Conditions 1, 2, and 3a then, following the proof of Theorem 1, we see that $f_i(v_R)$ depends only on r. So f can be written in the form of equation (4), where

$$k_i(t)=\sum_{r=t}^{n}(-1)^{r-t}\binom{n-t}{r-t}\hat{f}_i(r) \text{ and } \hat{f}_i(r)=f_i(v_R) \text{ for } i\in R.$$

A useful tool for examining other weighted sums is the multilinear extension of a game, as defined in [7]. The multilinear extension of a game v is defined on the unit cube of dimension n by

$$f(x_1, \ldots, x_n)=\sum_{S\subset N}\{\prod_{i\in S}x_i\prod_{i\notin S}(1-x_i)\}v(S).$$

Note that if $(x_1, \ldots, x_n)$ is a vector of zeroes and ones, then $f(x)=v(S)$ where $i\in S$ if and only if $x_i=1$.

Differentiating with respect to x_i, we have

$$f_i(x_1, \ldots, x_n)=\sum_{\substack{S\subset N\\ i\in S}}\{\prod_{\substack{j\in S\\ j\neq i}}x_j\prod_{j\notin S}(1-x_j)\}[v(S)-v(S-i)].$$

Owen observed in [8] that the extended Banzhaf-Coleman index is obtained by evaluating the gradient of f at the point $(\frac{1}{2}, \ldots, \frac{1}{2})$. In fact, we see that at every point the gradient of f is a weighted sum of terms of the form $[v(S)-v(S-i)]$. So by Theorem 2, we have the following result.

COROLLARY 1: The gradient of the multilinear extension of a game everywhere satisfies Conditions 1, 2, and 3a.

In [11] it was shown that the Shapley value is a von Neumann-Morgenstern utility function, characterized by the fact that $\phi_i(v_R)=\frac{1}{r}$ for $i \epsilon R$. It can be shown that every function f which satisfies Conditions 1, 2, and 3a is also a utility, characterized by the numbers $f_i(v_R)$. The author hopes to explore this further in a subsequent article [9].

REFERENCES

[1] Banzhaf, John F., III, "Weighted Voting Doesn't Work: A Mathematical Analysis," Rutgers Law Review, *19*, 317–343 (1965).

[2] Banzhaf, John F., III, "One Man, 3.312 Votes: A Mathematical Analysis of the Electoral College," Villanova Law Review, *13*, 304–332 (1968).

[3] Blair, Douglas H., "Power Valuations of Finite Voting Games," Presented at Econometric Society Winter Meeting (1975).

[4] Coleman, James S., "Control of Collectivities and the Power of a Collectivity to Act," *Social Choice*, B. Lieberman, ed., (Gordon and Breach, New York, 1971).

[5] Dubey, Pradeep, "Some Results on Values of Finite and Infinite Games," Ph.D. thesis, Cornell University (1975).

[6] Lucas, William F., "Measuring Power in Weighted Voting Systems," Technical Report No. 227, Department of Operations Research, Cornell University (1974).

[7] Owen, Guillermo, "Multi-Linear Extensions of Games," Management Science, *18*, 64–69 (1972).

[8] Owen, Guillermo, "Multilinear Extensions and the Banzhaf Value," Naval Research Logistics Quarterly, *22*, 741–750 (1975).

[9] Roth, Alvin E., "Bargaining Ability, the Utility of Playing a Game, and Models of Coalition Formation" (Mimeo, 1976).

[10] Roth, Alvin E., "Utility Functions for Simple Games," Working Paper No. 305, College of Commerce, University of Illinois (1976).

[11] Roth, Alvin E., "The Shapley Value as a von Neumann-Morgenstern Utility," Econometrica (forthcoming).

[12] Shapley, Lloyd S., "A Value for n-Person Games," Annals of Mathematics Study, *28*,307–317 (Princeton University Press, 1953).

☆U.S. GOVERNMENT PRINTING OFFICE: 1977 261-252/3 1-3

INFORMATION FOR CONTRIBUTORS

The NAVAL RESEARCH LOGISTICS QUARTERLY is devoted to the dissemination of scientific information in logistics and will publish research and expository papers, including those in certain areas of mathematics, statistics, and economics, relevant to the over-all effort to improve the efficiency and effectiveness of logistics operations.

Manuscripts and other items for publication should be sent to The Managing Editor, NAVAL RESEARCH LOGISTICS QUARTERLY, Office of Naval Research, Arlington, Va. 22217. Each manuscript which is considered to be suitable material for the QUARTERLY is sent to one or more referees.

Manuscripts submitted for publication should be typewritten, double-spaced, and the author should retain a copy. Refereeing may be expedited if an extra copy of the manuscript is submitted with the original.

A short abstract (not over 400 words) should accompany each manuscript. This will appear at the head of the published paper in the QUARTERLY.

There is no authorization for compensation to authors for papers which have been accepted for publication. Authors will receive 250 reprints of their published papers.

Readers are invited to submit to the Managing Editor items of general interest in the field of logistics, for possible publication in the NEWS AND MEMORANDA or NOTES sections of the QUARTERLY.

NAVAL RESEARCH LOGISTICS QUARTERLY

SEPTEMBER 197
VOL. 24, NO. 3
NAVSO P-1278

CONTENTS

ARTICLES

		Pa
Optimal Disposal Policies for a Single-Item Inventory System with Returns	D. P. HEYMAN	38
Compound Distributions with Efficient Computation in Inventory Model Applications	D. L. BOTT	40
Probability Density Function of a Stochastic Linear Programming Problem	W. S. HSIA	41
Sequential Tests for Hypotheses Concerning the Product of Poisson Parameters with Applications to Problems in Reliability and Biometry	B. HARRIS A. P. SOMS	42
Relation between Continuous and Discrete Time Markovian Decision Problems	P. KAKUMANU	43
A Nonconvex Max-Min Problem	J. E. FALK K. HOFFMAN	44
Single Server Queues with Hyperexponential Service Times	I. GREENBERG	45
Probabilistic Priorities in the M/G/1 Queue	D. E. MATTHEWS	45
Characterization of the Nucleolus for a Class of n-Person Games	G. OWEN	46
A Parallel Sequencing Algorithm for Minimizing Total Cost	M. K. RAJARAMAN	47
Special Cases of the Flow-Shop Problem	W. SZWARC	48
Adaptive Forecasting of the Size of a Work Force Subject to Random Withdrawals	S. ZACKS S. E. HABER	49
A Model for the Determination of Optimal Interfiring Times	Y. FRIEDMAN	50
A Note on the Relationship between Correlation and Circular Error Probability (CEP)	D. E. OLSEN	51
A Note on Values and Multilinear Extensions	A. E. ROTH	51

OFFICE OF NAVAL RESEARCH
Arlington, Va. 22217